Exam in Graph Theory, 5 January 2024 · 1MA170

Vilhelm Agdur[1]

5 January 2024

There are a total of ten questions on this exam. You are to **pick eight of them to answer**.[2]

Good luck!

[1] I will visit the exam hall at some point during the exam to answer any questions. If you need to reach me outside that time, I am available by email at `vilhelm.agdur@math.uu.se` or by phone at `072-373 32 90`.

[2] If you answer more than eight questions, your exam total score will be the sum of your eight *lowest* scores, so there is absolutely **no** benefit to answering more than eight questions.

Question 1 (5p)

What does it mean for a graph to be k-connected? State a correct definition.

We proved a characterisation of two-connected graphs as being exactly the graphs that can be constructed by a certain process starting from a cycle graph. State the theorem, and give a proof of it.

Question 2 (5p)

Define the adjacency matrix A, incidence matrix D, and Laplacian matrix Q of a graph. Prove that $Q = DD^t$.

Question 3 (5p)

Recall that Hall's marriage theorem gives a condition for the existence of a matching of one side of a bipartite graph into the other in terms of a condition on the size of sets $Q \subseteq A$ and their neighbour sets $N(Q) \subseteq B$.

Give a precise statement of the theorem, and then give a proof.[3]

[3] The proof we gave in the lecture was a clever application of the max-flow min-cut theorem, where we turned our bipartite graph into a flow network. If you choose to give this proof, please also draw a figure of the construction.

Question 4 (5p)

Part a: Prove that if you remove k edges from a tree T,[4] the resulting graph is a forest of $k+1$ trees.

[4] Where the tree has more than k vertices, so there are indeed k edges to remove.

Part b: Pick one of Kruskal's or Prim's algorithms. State it and prove its correctness.

Question 5 (5p)

What does it mean for a graph to be planar? What is the planar dual of a planar graph? Give definitions, and include a correct figure illustrating the definition.

Prove the following result from the course:[5]

Theorem 1 (Euler's formula). *Let $G = (V, E)$ be a connected planar graph, and let f be the number of faces for some planar embedding of G. Then*

$$|V| - |E| + f = 2,$$

and so in particular any two planar embeddings have the same number of faces.

[5] Hint: Consider a spanning tree of G and its complement in G^*.

Question 6 (5p)

What does it mean for a graph to be Eulerian? State and prove Euler's theorem giving a criterion for a graph to be Eulerian.

Question 7 (5p)

Recall from the lectures that the *minimum bisection problem* asks for a partition of the vertices of a graph into two equally-sized sets[6] such that the number of edges between them is minimal.

We proved an upper bound on how many edges you could be forced to include in such a bisection using the probabilistic method, where we started by picking a matching on the complement graph. State and prove this result.

[6] So, as we state it, it only applies to graphs with an even number of vertices.

Question 8 (5p)

Given a graph H and an integer n, define the extremal function $\mathrm{ex}(n; H)$ of H.

Turáns theorem says that

$$\mathrm{ex}(n; K_{r+1}) \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2}.$$

Prove this.[7]

[7] We gave two proofs in the course, and there are many more pretty proofs floating around. Any correct proof is of course a correct answer to the question.

The first proof we gave in the course used the Caro-Wei result on independent sets, and then applied the Cauchy-Schwarz inequality

$$|\langle a, b\rangle| \leq \|a\| \, \|b\|$$

to a clever choice of a and b.

Question 9 (5p)

A *directed acyclic graph* (often called just a *DAG*) is a directed graph that contains no cycles. Devise a reasonable[8] algorithm for determining when such a graph is Hamiltonian. For directed graphs, being Hamiltonian means having a *directed* Hamiltonian path.

[8] That is, not just a brute-force method, but something you'd actually use in practice.

Question 10 (5p)

For each of these statements, determine if it is true or false and give a proof or disproof:[9]

[9] Each statement gives one point, except for c) and d), which give half a point each.

a) For every $k \geq 3$, there exists a k-regular Hamiltonian graph.

b) For every $k \geq 1$, there exists a k-regular planar graph.

c) For every $k \geq 1$, there exists a multigraph with exactly k spanning trees.

d) For every $k \geq 1$, there exists a graph with exactly k spanning trees.

e) There exists a 6-connected planar graph.

f) There exists a tree in which every vertex has degree 2.