Recent Announcements

Working Groups Created (https://uppsala.instructure.com/courses/24243/discussion_topics/58020)
Good afternoon everyone! Working groups for assignment 1 have been creat...

Posted on: Feb 4, 2021 at 3:53pm

Mistake fixed in assignment 1. (https://uppsala.instructure.com/courses/24243/discussion_topics/57704)
Good afternoon everyone! As pointed out by one of the students, there was ...

Posted on: Feb 3, 2021 at 2:01pm

Assignment 1 available (https://uppsala.instructure.com/courses/24243/discussion_topics/57621)
Good morning everyone! Assignment 1 is now available. Instead of just givin...

Posted on: Feb 3, 2021 at 10:03am

Kombinatorik VT2021 (Period 3)

Jump to Today ✎ Edit

Welcome to the the combinatorics course!

The **course material** is available in the Modules (Moduler) section. There you will find suggested exercises, course lectures, as well as solutions to certain exercises.

The **zoom link** for the course lectures and problem sessions is:
(https://uu-se.zoom.us/j/62746932845) https://uu-se.zoom.us/j/69079713853 (https://uu-se.zoom.us/j/69079713853?pwd=bmJGekM3b21JTS9FVFBEeUliUWhxUT09)
password: factorial

The lectures will **not** be recorded, and so you are very encouraged to join. The lecture notes will be posted in the proper module afterwards. You are always welcome to ask questions during the lectures. I will always stick around for a while after the lectures if you have questions as well.

The **course textbook** is *Applied Combinatorics* by Keller and Trotter, and is available for free here:
https://rellek.net/book/app-comb.html (https://rellek.net/book/app-comb.html)
The solutions to some of the exercises from the textbook can be found here:
https://people.math.gatech.edu/~trotter/math-3012/toppage.html
(https://people.math.gatech.edu/~trotter/math-3012/toppage.html)

Grading: The exam on March 16 will be graded out of 40 points. You need:
18 points for a grade of 3
25 points for a grade of 4
32 points for a grade of 5

There will also be **3 assignments,** each made available following a problem session (see the schedule below). Once made available, you will have 2 days to complete the assignment. Each assignment will consist of two questions, each worth 5 points (10 points per assignment). These assignments are **not mandatory**, but they will be graded like exam questions so that you know how your exam will be graded. To encourage you to complete the assignment, extra points will be awarded towards your exam for completing assignments. The sum of your grades on the assignment determine the extra points for the exam:
15/30 -> 1 extra point
20/30 -> 2 extra points
25/30 -> 3 extra points
Each student must submit individual assignments, but you **very encouraged to work together** on the assignments and to ask me questions and for hints towards solutions

If you have any questions about any exercise you are working on or about anything at all, please send me an email at:
colin.desmarais@math.uu.se (mailto:colin.desmarai@math.uu.se)
I will try to always answer very quickly. If you prefer we can always try to set up a zoom meeting to answer your questions.

Course Schedule:

Module 1: Fundamental Principles

Jan. 19: Permutations + Combinations

Jan. 28: Combinatorial Proofs + Binomial Theorem

Jan. 29: Multinomial Coefficients + Distributions + Lattice Paths

Feb. 3: Problem Session 1

Module 2: Three Principles

Feb. 4: Review of Principle of Mathematical Induction + Principle of Inclusion/Exclusion

Feb. 8: Principle of Inclusion/Exclusion (continued)

Feb. 10: Pigeonhole Principle

Feb. 15: Problem Session 2

Module 3: Generating Functions and Recurrence Relations

Feb. 16: Generating Functions

Feb. 23: Recurrence Relations

Feb. 25: Further Examples

Mar. 3: Problem Session 3

Module 4: Discrete Probability

Mar. 4: Discrete Probability 1

Mar. 9: Discrete Probability 2

Exam Review

Mar. 10: Exam Review

Course Summary:

Date	Details	Due
Wed Feb 10, 2021	Assignment 1 (https://uppsala.instructure.com/courses/24243/assignments/49332)	due by 11:59pm
Mon Feb 22, 2021	Assignment 2 (https://uppsala.instructure.com/courses/24243/assignments/52453)	due by 11:59pm
Wed Mar 10, 2021	Assignment 3 (https://uppsala.instructure.com/courses/24243/assignments/54316)	due by 11:59pm
Tue Mar 16, 2021	Final Exam (https://uppsala.instructure.com/courses/24243/assignments/54822)	due by 1:20pm

Fundamental Principles

1.1 Permutations + Combinations

Textbook readings

- From Keller + Trotter: Section 2.2 – 2.3

Notation, Definitions, and Theorems

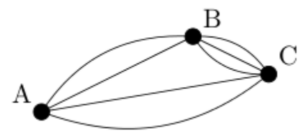
- **rule of sum** or **addition principle**: If the sets A and B share no common elements, then the number of ways of choosing something from A **or** choosing something from B is given by $|A \cup B| = |A| + |B|$.
- **rule of product** or **multiplications principle**: The number of ways of choosing something from a set A **and** choosing something from a set B is given by $|A \times B| = |A| \cdot |B|$.
- $n! := n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$
- $0! = 1$
- For nonnegative integers $n \geq k$, the number of ways of permuting k objects from a set of n objects is denoted
$$P(n,k) := \frac{n!}{(n-k)!}.$$
- A combination of size k from a set A of size n is a subset of A with k elements
- For nonnegative integers $n \geq k$, the number of combinations of size k from a set of n objects is given by
$$C(n,k) = \binom{n}{k} := \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}.$$
- $\binom{n}{k}$ is called a binomial coefficient

Exercises

Suggested exercises from textbook

- From Keller + Trotter: Section 2.9, exercises 1–3, 5, 6, 12, 13

Exercise 1.1.1. Below is a map of towns A, B, and C. There are 2 routes from A to B, 3 routes from B to C, and 2 direct route from A to C.



(a) How many ways are there of getting from A to C through B?

(b) What is the total number of ways of getting from A to C?

(c) How many ways are there of getting from A to C and then back to A?

Exercise 1.1.2. In Sweden, vehicle licence plates are made up of either: 3 letters followed by 3 numbers OR 3 letters followed by 2 numbers followed by 1 letter.

(a) What is the total number of possible licence plates?

(b) How many possible licence plates start with the letter S?

(c) How many possible licence plates end in 2?

(d) How many possible licence plates end in A?

(e) How many possible licence plates do not contain the letters A, B, C, or D?

Exercise 1.1.3. In the town near your summer house, there is an ice cream shop with 15 different flavours.

(a) In how many different ways can you try a new flavour every day without repeating your choice?

(b) You really like chocolate ice cream, so you choose chocolate on days 1, 4, 7, 10, 13, and a different flavour without repetition on days 2,3,5,6,8,9,11,12,14,15. How many ways can you do this?

Exercise 1.1.4. There are 10 people lining up to take the bus.

(a) How many ways can the people line up?

(b) If Anders does not want to be first in line, now how many ways are there?

(c) If also Agnes does not want to be last, how many way are there of lining up?

Exercise 1.1.5. There are 8 customers waiting to be seated at a restaurant with only round tables.

(a) How many ways can the customers be seated at one table?

(b) How many ways can the customers be seated at 2 tables of 4 people? (Suppose that it matters which table the customers are seated at.)

Exercise 1.1.6. You are back at your favourite ice cream shop with the 15 flavours.

(a) In how many ways can you choose 3 flavours in a bowl?

(b) There are also 5 types of toppings; sprinkles, cookies, chocolate syrup, caramel syrup, and chocolate chips. How many ways are there of making an ice cream sunday with 3 flavours of ice cream and 2 toppings?

Exercise 1.1.7. There are 24 students that want to form teams to play innebandy.

(a) How many ways can they form 4 teams named A, B, C, D?

(b) How many ways can they form these teams if Axel and Maja cannot be on the same team?

1.2 Combinatorial Proofs + Binomial Theorem

Textbook readings

- From Keller + Trotter: Section 2.4 and 2.6

Notation, Definitions, and Theorems

- **Pascal's identity:** For $1 \leq k < n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Theorem 1.2.1 (Binomial Theorem). *For real variables x and y and nonnegative integer n, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Exercises

Suggested exercises from textbooks

- From Keller + Trotter: Section 2.9, exercises 20, 21, 22, 24, 26, 29, 30.

Exercise 1.2.1. A pizza restaurant offers $2n$ choices for toppings for the pizzas.

(a) How many pizzas with n different toppings can be made?

(b) Suppose n of the choices of toppings are vegetables and n of the choices of toppings are cheeses. For $0 \leq k \leq n$, how many pizzas can be made with exactly k different vegetable toppings and $n-k$ different cheese toppings?

(c) Using parts (a) and (b), give a combinatorial proof that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Exercise 1.2.2. Provide both an algebraic and a combianatorial proof that for all $n \geq k \geq m \geq 0$,

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Exercise 1.2.3. What is the coefficient of $x^4 y^3$ in the expansion of

(a) $(x+y)^7$?

(b) $(x^2+y)^5$?

(c) $(x+2y)^7$?

Exercise 1.2.4. Use the binomial theorem to prove that

(a)

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

(b)

$$3^n\binom{n}{0} - 3^{n-1}\binom{n}{1} + 3^{n-2}\binom{n}{2} + \cdots + (-1)^{n-1}3\binom{n}{n-1} + (-1)^n\binom{n}{n} = 2^n.$$

Exercise 1.2.5. Prove that for all $n \geq 1$,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

1.3 Multinomial Coefficients + Distributions + Lattice Paths

Textbook readings

- From Keller + Trotter: Section 2.5 and 2.7

Notation, Definitions, and Theorems

- For nonnegative integers $n, k_1, k_2, \ldots, k_r$ such that $k_1 + k_2 + \cdots + k_r = n$, the multinomial coefficient is denoted by

$$\binom{n}{k_1, k_2, \ldots, k_r} := \frac{n!}{k_1! k_2! \cdots k_r!}.$$

- The n'th Catalan number is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Exercises

Suggested exercises from textbooks

- From Keller + Trotter: Section 2.9, exercises 29–31.

Exercise 1.3.1. How many integer solutions are there to

$$x_1 + x_2 + x_3 = 32$$

if

(a) $x_1, x_2, x_3 \geq 0$?

(b) $x_1 \geq 3,\ x_2 \geq 5,\ x_3 \geq 7$?

(c) $x_1, x_2 \geq 5,\ 0 \leq x_3 \leq 20$?

Exercise 1.3.2. How many ways can 20 kanelbullar be distributed amongst 4 students if

(a) there are no restrictions?

(b) every student gets at least one?

(c) the fourth student cannot have more than 10?

Exercise 1.3.3. How many ways can 12 apples and 7 muffins be distributed in 5 baskets if every basket must have at least 1 muffin?

Exercise 1.3.4. How many rearrangements of the letters of UPPSALA are there

(a) with no restrictions?

(b) that have no consecutive A's?

(c) that do not have U and S together?

Exercise 1.3.5. How many Up/Right paths are there from $(0, 0)$ to $(8, 10)$

(a) with no restrictions?

(b) that go through $(4, 7)$?

(c) that always take an even number of steps to the right? (for example, $RRURRRRUURR\ldots$ is allowed while $RRURUURRR\ldots$ is not.)

Exercise 1.3.6. Suppose you are trying to find n people to volunteer to clean-up a park. You carry with you a sign-up sheet and n pens in a bag, in case all are used at once. Because you like math, you keep track of the number of pens in the bag in a sequence. For example, if $n = 3$ and all are used at once, then the sequence would go $3, 2, 1, 0, 1, 2, 3$ as they pick a pen one-by-one and return it one-by-one. If 2 people came at first, and later in the day a third person, the sequence would go $3, 2, 1, 2, 3, 2, 3$.

(a) How many such sequences are there if $n = 3$?

(b) How many such sequences are there if $n = 4$?

(c) How many such sequences are there for any positive number n? (HINT: for every sequence, place a $\nearrow$ between two numbers if the sequence increases, and a $\searrow$ if it decreases, for example

$$3 \searrow 2 \searrow 1 \nearrow 2 \nearrow 3 \searrow 2 \nearrow 3.$$

What can you say about the sequence of arrows?)

Tuesday, January 19

Permutations + Combinations

What is Combinatorics?

- hard to define completely
- the mathematics of counting
- Some problems are purely combinatorial, others have combinatorial aspects.

Why combinatorics?

- Applications in Pure Mathematics
- Applications in Science
    - networks/graphs
    - Algorithm analysis
    - Biology
    - Physics
    - many more
- Puzzles: Easy to explain, surprising solutions.

Basic Principles

Rule of sum (addition principle): If A is a set of m different tasks, and B is a set of n other different tasks, then the number of ways of performing a task from A or a task from B is $m+n$.

Example: A restaurant has a menu consisting of 4 drink options, 10 main dishes, 5 side dishes, and 3 desserts.

How many items are on the menu?

$$4+10+5+3=22$$

Rule of product (multiplication principle): If A is a set of m different tasks, and B is a set of n different tasks, then the number of ways of performing a task from A and a task from B is $m \cdot n$.

Example: In the restaurant in the previous example, how many ways are there of ordering a meal consisting of a drink, a main course, a side dish, and a dessert?

$$4 \cdot 10 \cdot 5 \cdot 3 = 600$$

Strings:

Definition: A string (or word) of length n from the set X (called an alphabet) is a function $s:\{1,2,\ldots,n\} \to X$, where $s(i)$ is the i'th character. We usually write s as $s = x_1x_2x_3\cdots x_n$, where $x_i = s(i)$.

Example (binary strings): Let $X = \{0,1\}$. Strings $s:\{1,2,\ldots,n\} \to X$ are called binary strings. For example, here are the 8 binary strings of length 3

000	010	100	110
001	011	101	111

There are 2^n binary strings of length n:

For each i, there are 2 choices for $s(i)$: 0 or 1. So by the rule of product there are

$$\underbrace{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ times}} = 2^n$$

ways of forming a binary string of length n.

Example: (m-ary strings) Let $X = \{0,1,\ldots,m-1\}$. Then $s:\{1,2,\ldots,n\} \to X$ is a m-ary string of length n (binary if $m=2$, ternary if $m=3$)

There are m^n m-ary strings of length n.

For general set X, we call $s:\{1,2,\ldots,n\} \to X$ a X-string

Permutations:

Definition: A string $s:\{1,2,\dots,K\}\to X$, say $s = x_1x_2\cdots x_K$, is called a permutation of length K of the elements of X if all $x_1, x_2, \dots, x_K$ are different.

If $K \geq 1$, then clearly we need $|X| \geq K$ for a permutation of length K to exist.

Example: Let $X = \{1,2,3\}$. There are 6 permutations of length 2 of the elements of X:

12	21	31
13	23	32

There are 3 ways of choosing x_1 from X, and 2 ways of choosing x_2 from $X\setminus\{x_1\}$, so by the rule of product $3\cdot 2 = 6$ ways of forming a permutation of length 2.

Definition: For $n = 1,2,\dots,$ define $n! = n\cdot(n-1)\cdot(n-2)\cdot \cdots \cdot 3\cdot 2\cdot 1$.
Define $0! = 1$

For $n \geq K$, define $P(n,K) = \dfrac{n!}{(n-K)!}$

Notice that $P(n,n) = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$

Proposition: If $|X| = n$ and $0 \leq K \leq n$, then there are $P(n,K)$ permutations of length K from X.

Proof: There are n ways of choosing x_1

There are $|X\setminus\{x_1\}| = n-1$ ways of choosing x_2

There are $|X\setminus\{x_1,x_2\}| = n-2$ ways of choosing x_3

$\vdots$

There are $|X\setminus\{x_1,x_2,\dots,x_{K-1}\}| = n-(K-1) = n-K+1$ ways of choosing x_K.

By the rule of product, there are

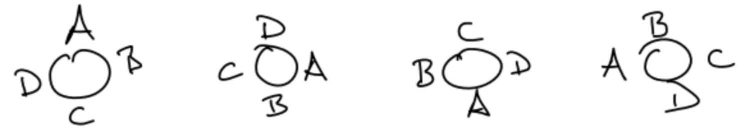
$$n\cdot(n-1)\cdot(n-2)\cdot \cdots \cdot (n-K+1) = \frac{n\cdot(n-1)\cdot \cdots \cdot (n-K+1)\cdot(n-K)\cdot \cdots \cdot 3\cdot 2\cdot 1}{(n-K)\cdot \cdots \cdot 3\cdot 2\cdot 1}$$

$$= \frac{n!}{(n-K)!} = P(n,K) \qquad \square$$

ways of forming a permutation of length K from X.

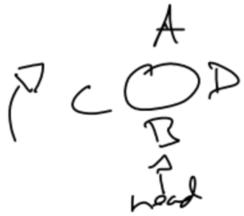
Example: How many ways can n people be seated at a round table?

In this example,



Count as the same seating of 4 people.

Let m be the number of ways of seating n people at a round table, let h be the number of ways of choosing a head of the table. Clearly $h=n$. If we seat n people, then choose a head of the table, then read the names from the head then going clockwise, we get a permutation of the names



$\longrightarrow$ BCAD

So by the rule of product, $m \cdot h = P(n,n)$

So the number of ways of seating n people at a round table is

$$m = \frac{P(n,n)}{h} = \frac{n!}{n} = \frac{\cancel{n}\cdot(n-1)\cdot \ldots \cdot 3\cdot 2\cdot 1}{\cancel{n}} = (n-1)!$$

Combinations

Definition: For a set X, a combination of elements from X is a subset $A \subseteq X$.

Example: There are 6 combinations of size 2 from $X = \{a, b, c, d\}$,

$\{a,b\}$ $\{a,d\}$ $\{b,d\}$

$\{a,c\}$ $\{b,c\}$ $\{c,d\}$.

Let $C(n,k)$ for the number of combinations of size k from X, $|X| = n$

Definition: For $0 \le k \le n$, let $\binom{n}{k} := \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$

The notation $C(n,k)$ or nCk or C^n_k is also often used

$\binom{n}{k}$:= n choose k, binomial coefficient.

Proposition: If $|X| = n$ and $0 \le k \le n$, then there are $\binom{n}{k}$ combinations of size k from X.

Proof: Let's look at $P(n,k)$ again. To make a permutation of size k from X, we could first choose a subset $A \subseteq X$ of size k, then make a permutation from A. There are $C(n,k)$ subsets A of size k from X, and $P(k,k) = k!$ permutations of the elements of A. So by the product rule,

$$C(n,k) \cdot k! = P(n,k)$$

$$\Rightarrow C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

$\square$

Proposition: $\binom{n}{k} = \binom{n}{n-k}$

Proof 1: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$ $\square$

Proof 2: Let $|X| = n$. Every time we choose a subset $A \subseteq X$ of size k, we are left with a subset $X \setminus A \subseteq X$ of size $n-k$, and every subset of size $n-k$ can be achieved this way. So,

$$\binom{n}{k} = C(n,k) = C(n,n-k) = \binom{n}{n-k} \quad \square$$

The first proof is algebraic, the second is combinatorial.

Thursday, January 28

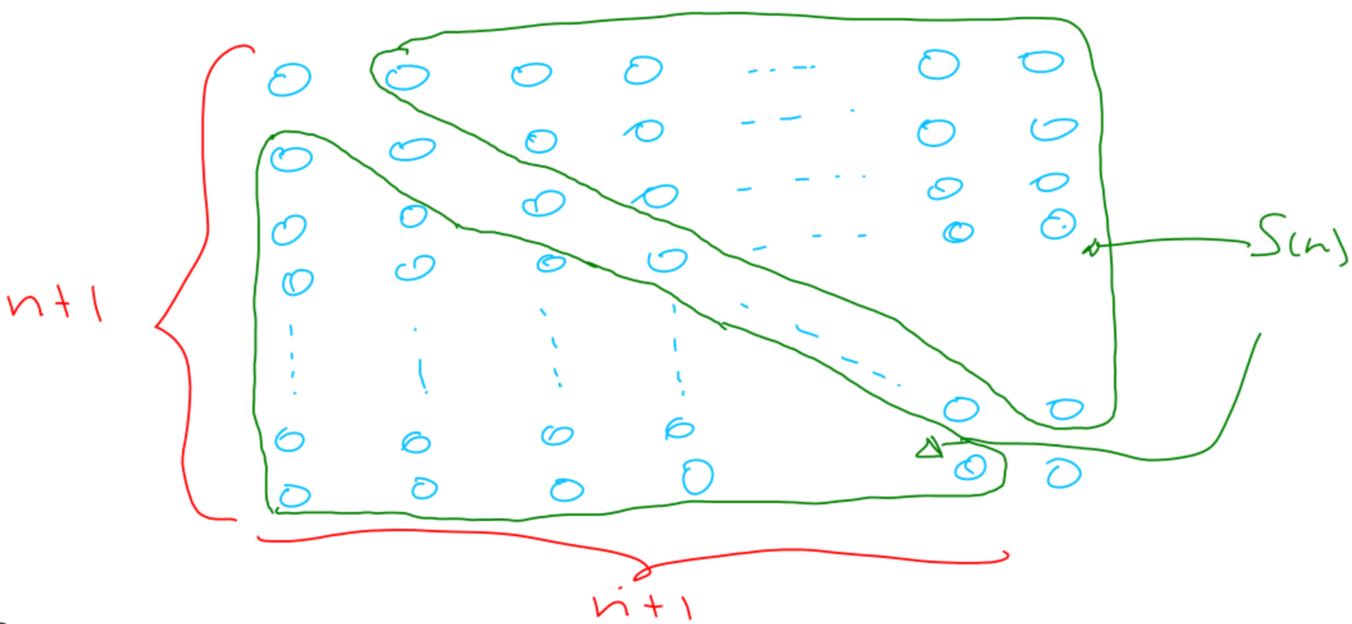
Combinatorial Proofs + Binomial Theorem

Combinatorial Proofs

The idea of a combinatorial proof is to provide a counting argument for some identity, often by counting the same thing in 2 different ways.

Example: For $n \geq 1$, let $S(n) = \sum_{k=1}^{n} k$. Prove that $S(n) = \frac{n(n+1)}{2}$.

Look at the $(n+1) \times (n+1)$ array below. Clearly there are $(n+1)\cdot(n+1) = (n+1)^2$ points.



Count the last n points in the first column, the last $n-1$ points in the second column, etc... the last point in the n'th column. Adding these points together gives $S(n) = \sum_{k=1}^{n} k$. There are also $S(n)$ points in the first point of the second column, first 2 points of the third column, etc... the first n points of the $(n+1)$'th column. There are $n+1$ points remaining on the diagonal. So, $2S(n) + (n+1) = (n+1)^2$

$$S(n) = \frac{(n+1)^2 - (n+1)}{2} = \frac{(n+1)((n+1)-1)}{2} = \frac{n(n+1)}{2}.$$

<u>Example</u>: Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

We saw last time that there are 2^n binary strings of length n.

Fix k between 0 and n, and count the number of binary strings with k 1's. Choose k positions for the 1's, and place 0's everywhere else

ex: $n=7$, $k=4$

$1\ 0\ 0\ 1\ 1\ 0\ 1$

$\{1,4,5,7\} \subseteq \{1,2,\ldots,7\}$

There are $\binom{n}{k}$ ways of choosing the k positions, so there are $\binom{n}{k}$ binary strings with k 1's.

Summing over all k, we count all binary strings. So

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

<u>Note</u>: Recall that a combination is a subset of X. So from above, we can also say that if $|X|=n$, then it has 2^n total subsets.

Pascal's identity: For $1 \le k \le n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

1	$n=0$
1 1	$n=1$
1 2 1	$n=2$
1 3 3 1	$n=3$
1 4 6 4 1	$n=4$
$k=0$ $k=1$ $k=2$ $k=3$ $k=4$	

Algebraic proof:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-k-1)!}$$

$$= \frac{k(n-1)!}{k!\,(n-k)!} + \frac{(n-k)(n-1)!}{k!\,(n-k)!}$$

$k(k-1)! = k!$ $\qquad$ $(n-k)(n-k-1)! = (n-k)!$

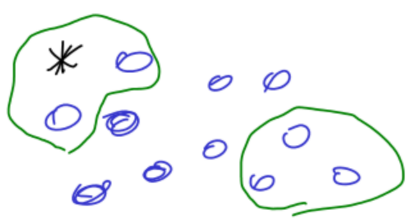
$$= \frac{(\cancel{k} + (n-\cancel{k}))(n-1)!}{k!\,(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!}$$

$n(n-1)! = n!$

$$= \binom{n}{k}$$

Combinatorial Proof: Look at a set X with n objects, and label one of the objects $*$. We know that X has $\binom{n}{k}$ subsets of size k.



To count the number of subsets of size k that include $*$, there are $\binom{n-1}{k-1}$ ways of choosing the remaining $k-1$ elements.

To count the number of subsets of size k that do not include $*$. There are $\binom{n-1}{k}$ ways of choosing the k elements from those that are not $*$. Altogether, there are $\binom{n-1}{k-1} + \binom{n-1}{k}$ subsets of size k.

So, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Binomial Theorem

Theorem: For any real numbers x and y, and for every $n \geq 0$,

$$\underbrace{(x+y)^n}_{\text{binomial}} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

$\binom{n}{k}$ ← binomial coefficients.

For example, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$$= \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3$$

$(x+y)^3 = (x+y)(x+y)(x+y)$ ← the $\binom{3}{1} = 3$ ways of getting x^2y

Proof of the binomial Theorem:

Look at $(x+y)^n = \underbrace{(x+y)(x+y) \cdots\cdots (x+y)}_{n \text{ factors}}$

If we multiply everything out, we are left with $\{x,y\}$-strings of length n as the terms of the expansion, each term constructed by choosing x or y in each factor

$(x+y)(x+y)(x+y) \cdots\cdots (x+y)(x+y)$ $\qquad$ $xxy \cdots\cdots yx$

For every k, once simplified, every string with k y's (and $n-k$ x's) simplifies to $x^{n-k}y^k$. Just as above with binary strings, there are $\binom{n}{k}$ $\{x,y\}$-strings with k y's, so the coefficient of $x^{n-k}y^k$ after simplification is $\binom{n}{k}$.

This holds for all k, so $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

Example: Provide 2 different proofs that

$$3^n = \binom{n}{0}2^0 + \binom{n}{1}2^1 + \cdots + \binom{n}{n}2^n. \qquad \left[\text{Hint: recall } \binom{n}{k} = \binom{n}{n-k}\right]$$

Proof 1: We count ternary strings ({0,1,2}-strings) of length n.

For each position, there are 3 choices, so there are 3^n ternary strings of length n.

For k between 0 and n, we count ternary strings with exactly k 2's. There are $\binom{n}{k}$ ways of choosing the k positions for the 2's, and every other position has 2 choices, either 0 or 1. So the number of ternary strings of length n with exactly k 2's is given by $\binom{n}{k}2^{n-k} = \binom{n}{n-k}2^{n-k}$.

Summing over all k, we get the total number of ternary strings, so

$$3^n = \binom{n}{0}2^0 + \binom{n}{1}2^1 + \cdots + \binom{n}{n}2^n. \qquad \square$$

Proof 2: Use the Binomial Theorem

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k$$

$$= \binom{n}{0}2^0 + \binom{n}{1}2^1 + \cdots + \binom{n}{n}2^n \qquad \square$$

Friday, January 29

Multinomial coefficients + Distributions + Lattice Paths

Rearrangements (Part 1)

By a rearrangement of a X-string s, we mean another X-string s' where each element of X appears the same number of times in s and s'.

Example: There are now 6 rearrangements of ABBA:

ABBA AABB BABA

ABAB BAAB BBAA

Proposition: Let $X=\{a,b\}$ have two elements. The number X-strings of length n with k a's and $n-k$ b's is given by

$$\binom{n}{k}=\binom{n}{n-k}$$

Proof: Same as number of binary strings of length n with k 1's which we saw last time. Out of the n positions, choose k places for a's, the remaining places for b's, and there $\binom{n}{k}$ ways of making this choice $\square$

As a consequence, if s is a string of length n, consisting of 2 symbols say a and b, with k a's, then there are $\binom{n}{k}$ rearrangements of s.

Now we review some applications of rearrangements:

Distributions (compositions)

Suppose we have n indistinguishable objects (all the same) to be distributed amongst K distinguishable people.

Example: 3 students A, B, C are competing for 2 possible extra bonus for the exam. In how many ways can the points be distributed?

6 ways:

$n=2$

$K=3$

A	B	C
• •		
•	•	
•		•
	• •	
	•	•
		• •

Proposition: There are $\binom{n+K-1}{K-1} = \binom{n+K-1}{n}$ ways of distributing n indistinguishable objects amongst K distinguishable people.

Proof (bars and stars argument): Consider a rearrangement of $K-1$ bars $|$ and n stars $\ast$

ex: $K=3$ (2 bars), $n=2$ (2 stars)

$\ast \ast | |$

$\ast | \ast |$

$\ast | | \ast$

$| \ast \ast |$

$| \ast | \ast$

$| | \ast \ast$

ex: $K=6$ $n=9$

$$\overbrace{\ast\ast}^{1}|\overbrace{\ast}^{2}|\overbrace{\;}^{3}|\overbrace{\ast}^{4}|\overbrace{\ast\ast\ast}^{5}|\overbrace{\ast\ast}^{6}$$

Give the stars before the first bar to person 1, the stars between the first and second bars to person 2, etc... until person K gets the stars after the $(K-1)$'th bar. There are $\binom{n+K-1}{K-1} = \binom{n+K-1}{n}$ rearrangements. So there are $\binom{n+K-1}{K-1} = \binom{n+K-1}{n}$ ways of distributing the objects. $\square$

Example: There are how many integer solutions to

$$x_1 + x_2 + x_3 = 12, \quad x_1, x_2, x_3 \geq 0 ?$$

x_1, x_2, x_3 are integers.

$12 = \underbrace{1+1+1+\cdots+1}_{12}$

This is distributing 12 1's amongst 3 variables. There are $\binom{12+3-1}{3-1} = \binom{14}{2}$ such distributions.

Example: There are how many integer solutions to

$$x_1 + x_2 + x_3 = 12, \quad x_1 \geq 1, \; x_2 \geq 3, \; x_3 \geq 5 ?$$

Start by giving one 1 to x_1, 3 1's to x_2, 5 1's to x_3. There are now $12 - (1+3+5) = 3$ 1's remaining, and $\binom{3+3-1}{3-1} = \binom{5}{2}$ ways of distributing what's left.

Or Let $y_1 = x_1 - 1$, $y_2 = x_2 - 3$, $y_3 = x_3 - 5$, then solutions $y_1 + y_2 + y_3 = 3$, $y_1, y_2, y_3 \geq 0$, are solutions to $3 + 9 = y_1 + y_2 + y_3 + 1 + 3 + 5 = y_1 + 1 + y_2 + 3 + y_3 + 5 = x_1 + x_2 + x_3$, $x_1 \geq 1$, $x_2 \geq 3$, $x_3 \geq 5$.

Lattice Paths

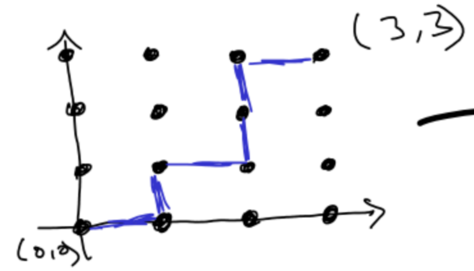
Paths on the integer lattice $\mathbb{Z} \times \mathbb{Z}$ consisting of

$\{(a,b) : a \in \mathbb{Z}, b \in \mathbb{Z}\}$

R) movement to the right $(x,y) \mapsto (x+1, y)$, or

U) movement up $(x,y) \mapsto (x, y+1)$

Example:



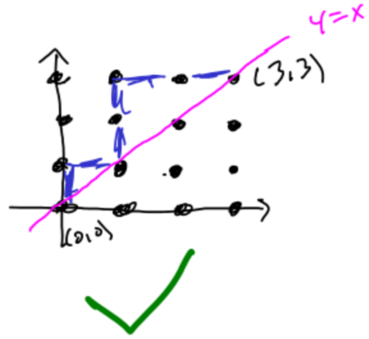
$\longrightarrow$ RURUUR

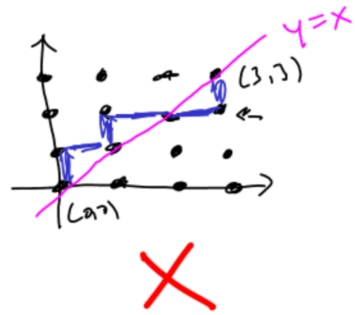
How many lattice paths are there from $(0,0)$ to $(3,3)$? Any path will consist of 3 R movements and 3 U movements. So we can look at all rearrangements of RRRUUU. There are $\binom{6}{3}$ rearrangements, so $\binom{6}{3}$ such paths.

In general, there are $\binom{m+n}{m} = \binom{m+n}{n}$ lattice paths from $(0,0)$ to (m,n).

Catalan numbers

There are how many lattice paths from $(0,0)$ to (n,n) that never go below the line $y=x$?





Definition: for a positive integer n, define the Catalan number

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

Proposition: The number of lattice paths from $(0,0)$ to (n,n) that never go below the line $y=x$ is $C(n) = \frac{1}{n+1}\binom{2n}{n}$.

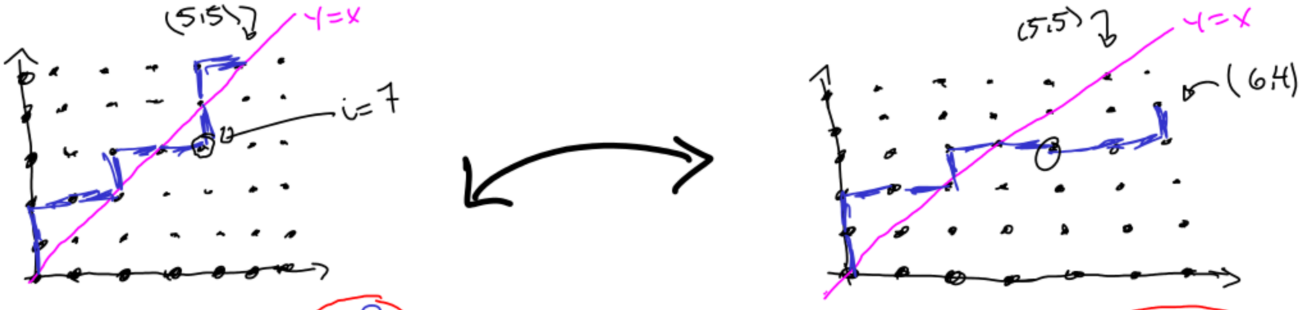
Proof: Let G be the set of "good" paths that never go below $y=x$, and B the set of "bad" paths that go below $y=x$. We want $|G|$.

Clearly $|G|+|B| = \binom{2n}{n}$, the total # of paths from $(0,0)$ to (n,n). Each path corresponds to a rearrangement of $\underbrace{UUU\cdots U}_{n}\underbrace{RRR\cdots R}_{n}$. A path is "good", if for all $1 \le i \le n$, there are the same number or more U's than R's in the first i positions.

ex: URUURR is "good", while $\underset{1\,2\,3\,4\,5\,6}{URURRU}$ is "bad" since in the first 5 positions there are more R's than U's

Take a bad path s, and let i be the smallest number for which there are more R's than U's in the first i positions. Say there are t U's and $t+1$ R's in the first i positions. For every position $j>i$, replace the remaining $n-t$ remaing U's with R's and replace the remaining $n-(t+1)$ R's with U's. There are now $t+n-(t+1) = n-1$ U's and $t+1+(n-t) = n+1$ R's. So the path s is now a path from $(0,0)$ to $(n+1, n-1)$.

ex:



UURRURRUUR $\qquad$ UURRURRRRU

$\underbrace{\text{UURRURR}}_{4 \text{ R's},\ 3 \text{ U's}}$ $\quad$ i=7

In the other direction, if we take a path from $(0,0)$ to $(n+1,n-1)$, there must eventually be more R's than U's in the first i positions. At that point, change the remaining R's to U's and U's to R's to get a "bad" path from $(0,0)$ to (n,n).

So $|B|$ is equal to the number of paths from $(0,0)$ to $(n+1,n-1)$ which is $\binom{(n+1)+(n-1)}{n-1} = \binom{2n}{n-1}$.

Therefore,
$$
\begin{aligned}
|G| &= \binom{2n}{n} - |B| \\
&= \binom{2n}{n} - \binom{2n}{n-1} \\
&= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)!} \\
&= \frac{(n+1)(2n)!}{(n+1)!\,n!} - \frac{n\,(2n)!}{n!\,(n+1)!} \\
&= \frac{\cancel{n(2n)!} + (2n)! - \cancel{n(2n)!}}{(n+1)\,n!\,n!} = \frac{1}{n+1}\binom{2n}{n}
\end{aligned}
$$

$(n+1)n! = (n+1)!$ $\qquad$ $n(n-1)! = n!$

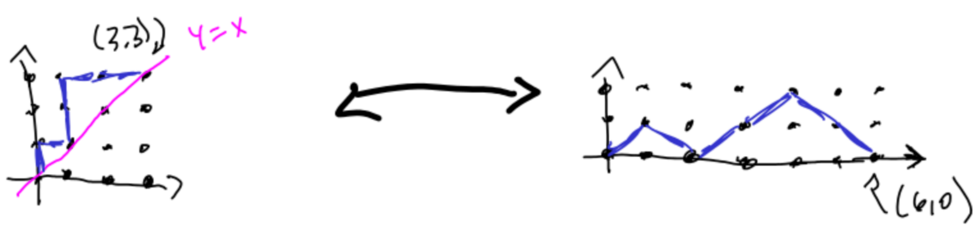
$\not\!\!\!\nearrow$

Note: A Dyck path is path from $(0,0)$ to $(2n,0)$ consisting of

U) $(x,y) \mapsto (x+1,y+1)$

D) $(x,y) \mapsto (x+1,y-1)$

that never goes below the line $x=0$. There are $C(n)$ Dyck paths from $(0,0)$ to $(2n,0)$



Rearrangements (Part 2) + multinomial coefficients

There are how many rearrangements of DATABASES?

DATABASES has 9 letters, 3 A's, 2 S's, and one each of DTBE. There are $\binom{9}{3}$ ways of choosing positions for A's, $\binom{6}{2}$ ways of choosing positions for S's from the remaining 6, $\binom{4}{1}$ ways of choosing positions for D

$\binom{3}{1}$ ——— // ——— for T

$\binom{2}{1}$ ——— // ——— for B

$\binom{1}{1}$ ——— // ——— for E.

So there are

$$\binom{9}{3}\binom{6}{2}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \frac{9!}{3!\cancel{6!}}\cdot\frac{\cancel{6!}}{2!\cancel{4!}}\cdot\frac{\cancel{4!}}{1!\cancel{3!}}\cdot\frac{\cancel{3!}}{1!\cancel{2!}}\cdot\frac{\cancel{2!}}{1!\cancel{1!}}\cdot\frac{\cancel{1!}}{1!\underbrace{0!}_{=1}} = \frac{9!}{\underset{A's}{3!}\,\underset{S's}{2!}\,\underset{D}{1!}\,\underset{T}{1!}\,\underset{B}{1!}\,\underset{E}{1!}} = \binom{9}{3,2,1,1,1,1}$$

Definition: For nonnegative integers $n, k_1, k_2, \ldots, k_r$ with $k_1+k_2+\cdots+k_r = n$, the multinomial coefficient is denoted

$$\binom{n}{k_1, k_2, \ldots, k_r} := \frac{n!}{k_1!k_2!\cdots k_r!}$$

Note: When $r=2$, we simply denote $\binom{n}{k_1,k_2}$ by the binomial coefficients $\binom{n}{k_1,k_2} = \binom{n}{k_1} = \binom{n}{k_2}$ (Since $k_1+k_2=n$)

Proposition: Suppose s is a X-string of length n consisting of k_1 x_1's, k_2 x_2's, $\ldots$, k_r x_r's with $k_1+\cdots+k_r=n$. There are $\binom{n}{k_1,\ldots,k_r} = \frac{n!}{k_1!\cdots k_r!}$ rearrangements of s.

We could prove this proposition by a similar argument as above, but let's look at a different argument.

Proof: Consider a new set

$$X' = \left\{\begin{matrix} x_1^1, x_1^2, \ldots, x_1^{k_1} \\ x_2^1, x_2^2, \ldots, x_2^{k_2} \\ \ldots \\ x_r^1, x_r^2, \ldots, x_r^{k_r} \end{matrix}\right\}$$

ex: $DA^1TA^2BA^3S^1ES^2$

Let R be the number of rearrangements of S. Let's count all permutations of length n from X'. To do this, take a rearrangement of S, and replace x_j's with $x_j^1, x_j^2, \dots, x_j^{k_j}$. There are $k_j!$ ways of replacing x_j's (# of permutations of $x_j^1, x_j^2, \dots, x_j^{k_j}$). By the product rule, the number of permutations of length n from X' is given by

$$R \cdot k_1! \, k_2! \cdots k_r! = n!$$

By rearranging, $R = \dfrac{n!}{k_1! k_2! \cdots k_r!} = \dbinom{n}{k_1, k_2, \dots, k_r}$

$\square$

Theorem (Multinomial Theorem): For any real numbers $x_1, x_2, \dots, x_r$, and any positive integer n,

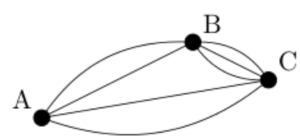
$$(x_1 + x_2 + \cdots + x_r)^n = \underbrace{\sum_{\substack{k_1, k_2, \dots, k_r \geq 0 \\ k_1 + k_2 + \cdots + k_r = n}}}_{} \binom{n}{k_1, k_2, \dots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}.$$

Sum over all possible integers $k_1 \geq 0, k_2 \geq 0, \dots, k_r \geq 0$ such that $k_1 + k_2 + \cdots + k_r = n$.

Solutions: Fundamental Principles

1.1 Permutations + Combinations

Exercise 1.1.1. Below is a map of towns A, B, and C. There are 2 routes from A to B, 3 routes from B to C, and 2 direct route from A to C.



(a) How many ways are there of getting from A to C through B?

(b) What is the total number of ways of getting from A to C?

(c) How many ways are there of getting from A to C and then back to A?

Solution. (a) By the rule of product, there are $2 \cdot 3 = 6$ ways.

(b) There are 2 direct routes from A to C, and 6 ways that go through B, so by the rule of sum, there are $2 + 6 = 8$ ways go getting from A to C.

(c) There are 8 ways from A to C, and 8 ways from C to A, so by the rule of product, there are $8 \cdot 8 = 64$ ways of getting from A to C and back to A.

□

Exercise 1.1.2. In Sweden, vehicle licence plates are made up of either: 3 letters followed by 3 numbers OR 3 letters followed by 2 numbers followed by 1 letter.

(a) What is the total number of possible licence plates?

(b) How many possible licence plates start with the letter S?

(c) How many possible licence plates end in 2?

(d) How many possible licence plates end in A?

(e) How many possible licence plates do not contain the letters A, B, C, or D?

Solution. (a) For the first option, by the rule of product there are

$$\underbrace{26 \cdot 26 \cdot 26}_{\text{letter}} \cdot \underbrace{10 \cdot 10 \cdot 10}_{\text{numbers}}$$

possibilities, and for the second option there are, by the rule of product,

$$\underbrace{26 \cdot 26 \cdot 26}_{\text{letter}} \cdot \underbrace{10 \cdot 10}_{\text{numbers}} \cdot \underbrace{26}_{\text{letter}}$$

possibilities. By the rule of sum, there are a total of

$$26^3 \cdot 10^3 + 26^4 \cdot 10^2 = 26^3 \cdot 10^2(26+10) = 26^3 \cdot 10^2 \cdot 36$$

possibilites.

(b) Fixing S as the first letter, there are

$$1 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 + 1 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 26 = 26^2 \cdot 10^2 \cdot 36$$

possibilities.

(c) Fixing 2 in the last position, there are

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 1 = 26^3 \cdot 10^2$$

possibilities.

(d) Fixing A in the last position, there are

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 1 = 26^3 \cdot 10^2$$

possibilities.

(e) Removing A, B, C, D leaves 22 possible letters. So the total number of possibilities is

$$22 \cdot 22 \cdot 22 \cdot 10 \cdot 10 \cdot 10 + 22 \cdot 22 \cdot 22 \cdot 10 \cdot 10 \cdot 22 = 22^2 \cdot 10^2 \cdot 32.$$

□

Exercise 1.1.3. In the town near your summer house, there is an ice cream shop with 15 different flavours.

(a) In how many different ways can you try a new flavour every day without repeating your choice?

(b) You really like chocolate ice cream, so you choose chocolate on days 1, 4, 7, 10, 13, and a different flavour without repetition on days 2,3,5,6,8,9,11,12,14,15. How many ways can you do this?

Solution. (a) This is just the number of permutations of the 15 flavours of which there are

$$P(15, 15) = 15!$$

(b) Removing days 1,4,7,10,13 leaves 10 days. Take a permutation of length 10 from the remaining 14 non-chocolate falvours, then insert a chocolate every three days. There are

$$P(14, 10) = \frac{14!}{4!}$$

ways of doing this.

□

Exercise 1.1.4. There are 10 people lining up to take the bus.

(a) How many ways can the people line up?

(b) If Anders does not want to be first in line, now how many ways are there?

(c) If also Agnes does not want to be last, how many way are there of lining up?

Solution. (a) This is the number of ways of permuting 10 people, which is

$$P(10, 10) = 10!$$

(b) Fixing Anders in the front and permuting the remaining 9 people, there are

$$P(9, 9) = 9!$$

permutations of the 10 people with Anders in front. If we let A be the number of permutations where Anders is not in the front, then by the rule of sum, $A + 9! = 10!$. Therefore, there are

$$A = 10! - 9!$$

permutations of teh 10 people where Anders is not at the front.

(c) There are 9! permutations where Agnes is at the back, and 9! permutations where Anders is at the front. If we look at

$$10! - 9! - 9!,$$

then we have removed certain permutations twice; the permutations where Anders is at the front AND Agnes is at the back have been removed twice. These need to be added back into the formula above. There are 8! permuations of the 10 people where Anders is at the front AND Agnes is at the back. Therefore, the number of permutations where Anders is not at the fton and Agnes is not at the back is given by

$$10! - 9! - 9! + 8!.$$

□

Exercise 1.1.5. There are 8 customers waiting to be seated at a restaurant with only round tables.

(a) How many ways can the customers be seated at one table?

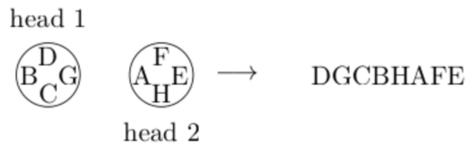
(b) How many ways can the customers be seated at 2 tables of 4 people? (Suppose that it matters which table the customers are seated at.)

Solution. (a) There are

$$\frac{P(8, 8)}{8} = \frac{8!}{8} = 7!$$

ways of seating the customers around the tables.

(b) If we seat the 8 customers, then choose a head of table 1 and a head of table 2, then list the customers starting at the head of table 1 and going clockwise then going to the head of table 2 and going clockwise, we get a permutation of the 8 customers.



Let h be the number of choosing the head of the tables. Then $h = 4 \cdot 4 = 16$. Let m be the number of ways of seating the 8 guests. Then by the rule of product, $m \cdot 8 = P(8,8) = 8!$. Therefore, the number of ways of seating the guests is

$$m = \frac{P(8,8)}{h} = \frac{8!}{16} = \frac{7!}{2}.$$

□

Exercise 1.1.6. You are back at your favourite ice cream shop with the 15 flavours.

(a) In how many ways can you choose 3 flavours in a bowl?

(b) There are also 5 types of toppings; sprinkles, cookies, chocolate syrup, caramel syrup, and chocolate chips. How many ways are there of making an ice cream sunday with 3 flavours of ice cream and 2 toppings?

Solution. (a) There are $\binom{15}{3}$ ways of choosing 3 flavours from a set of 15.

(b) There are $\binom{15}{3}$ ways of choosing 3 flavours, and $\binom{5}{2}$ ways of choosing 2 topping. By the rule of product, there are

$$\binom{15}{3}\binom{5}{2}$$

ways of making a Sunday.

□

Exercise 1.1.7. There are 24 students that want to form teams to play innebandy.

(a) How many ways can they form 4 teams named A, B, C, D?

(b) How many ways can they form these teams if Axel and Maja cannot be on the same team?

Solution. (a) Each team has $24/4 = 6$ players. There are $\binom{24}{6}$ ways of forming team A, $\binom{18}{6}$ ways of forming team B from the remaining players, $\binom{12}{6}$ ways of forming team C,and $\binom{6}{6}$ ways of forming team D. So altogether there are

$$\binom{24}{6}\binom{18}{6}\binom{12}{6}\binom{6}{6} = \frac{24!}{6!18!} \cdot \frac{18!}{6!12!} \cdot \frac{12!}{6!6!} \cdot \frac{6!}{6!0!} = \frac{24!}{6!6!6!6!}$$

ways of forming the 4 teams.

(b) If Axel and Maja are together in team A, there are $\binom{22}{4}$ ways of choosing the remaining 4 players, then $\binom{18}{6}$ ways of forming team B, $\binom{12}{6}$ ways of forming team C, and $\binom{6}{6}$ ways of forming team D. We repeat this argument 3 more times if Axel and Maja are on team B, C, or D. So altogether there are

$$4 \cdot \binom{22}{4}\binom{18}{6}\binom{12}{6}\binom{6}{6} = 4 \cdot \frac{22!}{4!18!} \cdot \frac{18!}{6!12!} \cdot \frac{12!}{6!6!} \cdot \frac{6!}{6!0!} = \frac{22!}{3!6!6!6!}$$

ways of forming teams with Axel and Maja together. Therefore there are

$$\frac{24!}{6!6!6!} - \frac{22!}{3!6!6!}$$

ways of forming teams where Axel and Maja are on the same team.

$\square$

1.2 Combinatorial Proofs + Binomial Theorem

Exercise 1.2.1. A pizza restaurant offers $2n$ choices for toppings for the pizzas.

(a) How many pizzas with n different toppings can be made?

(b) Suppose n of the choices of toppings are vegetables and n of the choices of toppings are cheeses. For $0 \leq k \leq n$, how many pizzas can be made with exactly k different vegetable toppings and $n-k$ different cheese toppings?

(c) Using parts (a) and (b), give a combinatorial proof that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Solution. (a) There are $\binom{2n}{n}$ ways of choosing n toppings, so $\binom{2n}{n}$ pizzas can be made.

(b) There are $\binom{n}{k}$ ways of choosing the vegetable toppings, and $\binom{n}{n-k}$ ways of choosing the cheese toppings, so by the rule of product there are

$$\binom{n}{k}\binom{n}{n-k}$$

pizzas that can be made with exactly k different vegetable toppings and $n-k$ different cheese toppings.

(c) Suppose a pizza restaurant has $2n$ different toppings, where n are vegetable toppings and n are cheese toppings. There are $\binom{2n}{n}$ different pizzas that can be made with n different toppings. We can also consider all the ways of making pizzas with k different vegetable toppings and $n-k$ different cheese toppings. There are $\binom{n}{k}\binom{n}{n-k}$ ways of making such pizzas. Summing over all $0 \leq k \leq n$, we recover the number of ways of making pizzas with n different toppings. Using the fact that $\binom{n}{k} = \binom{n}{n-k}$, we get

$$\begin{aligned}
\binom{2n}{n} &= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} \\
&= \sum_{k=0}^{n} \binom{n}{k}^2.
\end{aligned}$$

□

Exercise 1.2.2. Provide both an algebraic and a combianatorial proof that for all $n \geq k \geq m \geq 0$,

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Algebraic Proof: We use the definition of binomial coefficients.

$$\begin{aligned}\binom{n}{k}\binom{k}{m} &= \frac{n!}{k!(n-k)!}\frac{k!}{m!(k-m)!} \\ &= \frac{n!}{(n-k)!m!(k-m)!} \\ &= \frac{n!(n-m)!}{m!(n-m)!(n-k)!(k-m)!} \\ &= \frac{n!}{m!(n-m)!}\frac{(n-m)!}{(k-m)!((n-m)-(k-m))!} \\ &= \binom{n}{m}\binom{n-m}{k-m}.\end{aligned}$$

□

Combinatorial Proof: Suppose you visit a bakery with n pastries. You want to choose k pastries for you and your friends, where you will have m of those pastries. You could first choose k pastries, of which there are $\binom{n}{k}$ ways of doing, and then choosing m of those for yourself, which there are $\binom{k}{m}$ ways of doing, so $\binom{n}{k}\binom{k}{m}$ ways of making these choices. Alternatively, you could first choose m pastries for yourself, which you can do $\binom{n}{m}$ ways, and then choose the remaining $k-m$ pastries for you friends amongst the $n-k$ pastries remaining; there are $\binom{n-k}{k-m}$ ways of doing this. Altogether, this gives $\binom{n}{m}\binom{n-m}{k-m}$ ways of making the choices. Both methods give the same choices, so

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

□

Exercise 1.2.3. What is the coefficient of x^4y^3 in the expansion of

(a) $(x+y)^7$?

(b) $(x^2+y)^5$?

(c) $(x+2y)^7$?

Solution. We use the Binomial Theorem.

(a)

$$(x+y)^7 = \sum_{k=0}^{7}\binom{7}{k}x^{7-k}y^k,$$

so with $k=3$, the coefficient of x^4y^3 is $\binom{7}{3}$.

(b)

$$(x^2+y)^5 = \sum_{k=0}^{5}\binom{5}{k}(x^2)^{5-k}y^k,$$

so with $k=3$, the coefficient of $x^4y^3 = (x^2)^2y^3$ is $\binom{5}{3}$.

(c)

$$(x+2y)^7 = \sum_{k=0}^{7}\binom{7}{k}x^{7-k}(2y)^k = \sum_{k=0}^{7}2^k\binom{7}{k}x^{7-k}y^k,$$

so with $k=3$, the coefficient of x^4y^3 is $2^3\binom{7}{3}$.

$\square$

Exercise 1.2.4. Use the binomial theorem to prove that

(a)

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

(b)

$$3^n\binom{n}{0} - 3^{n-1}\binom{n}{1} + 3^{n-2}\binom{n}{2} + \cdots + (-1)^{n-1}3\binom{n}{n-1} + (-1)^n\binom{n}{n} = 2^n.$$

Solution. (a)

$$\begin{aligned}\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} &= \sum_{k=0}^{n}\binom{n}{k} \\ &= \sum_{k=0}^{n}\binom{n}{k}1^{n-k}1^k \\ &= (1+1)^n \\ &= 2^n\end{aligned}$$

(b)

$$\begin{aligned}3^n\binom{n}{0} - 3^{n-1}\binom{n}{1} + 3^{n-2}\binom{n}{2} + \cdots + (-1)^{n-1}3\binom{n}{n-1} + (-1)^n\binom{n}{n} &= \sum_{k=0}^{n}\binom{n}{k}3^{n-k}(-1)^k \\ &= (3+(-1))^n \\ &= 2^n.\end{aligned}$$

$\square$

Exercise 1.2.5. Prove that for all $n \geq 1$,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Solution 1. We can use the Binomal Theorem. We can see that

$$\begin{aligned}0 &= (1+(-1))^n \\ &= \sum_{k=0}^{n}\binom{n}{k}1^{n-k}(-1)^k \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots,\end{aligned}$$

which after bringing the negative terms in the last line to the left side of the equation gives

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

$\square$

Solution 2. We can use a combinatorial argument. Let A be a set with n elements.

First suppose n is odd. Then every time we choose an even number of elements of A to make a subset $B \subseteq A$, there are an odd number that remain in $A \setminus B$. So the total number of ways of forming a subset with an even number of elements is equal to the number of ways of forming a subset with an odd number of elements.

Now suppose n is even, and let a be an element of A. Let $C = A \setminus \{a\}$. Then C has an odd number of elements, and so has the same number of subsets with an even number of elements as subsets with an odd number of elements. These are also all the subsets of A that do not contain a, so there are the same number of even and odd subsets of A that do not contain a. Any even subset of A that contains a is an odd subset of C once we remove a, and every odd subset of A that contains a is an even subset of C once we remove a. So there are the same number of even subsets of A that contain a as odd subsets of A that contain a. Altogether, we get that A has the same number of even subsets and odd subsets.

The number of even subsets is given by

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

and the number of odd subsets is given by

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots,$$

and as we proved, these sums must be equal. $\square$

1.3 Multinomial Coefficients + Distributions + Lattice Paths

Exercise 1.3.1. How many integer solutions are there to

$$x_1 + x_2 + x_3 = 32$$

if

(a) $x_1, x_2, x_3 \geq 0$?

(b) $x_1 \geq 3,\ x_2 \geq 5,\ x_3 \geq 7$?

(c) $x_1, x_2 \geq 5,\ 0 \leq x_3 \leq 20$?

Solution. (a) This is the same as distributing 32 indistinguishable objects amongst 3 distinguishable people, so there are

$$\binom{32+3-1}{3-1} = \binom{34}{2}$$

solutions.

(b) First give 3 1's to x_1, 5 1's to x_2, and 7 1's to x_3. There are then $32 - (3+5+7) = 17$ remaining ones to distribute, which can be done

$$\binom{17+3-1}{3-1} = \binom{19}{2}$$

ways.

(c) First we find the number of solutions if $x_1, x_2 \geq 5$ and $x_3 \geq 0$. Start by giving 5 1's to x_1 and 5 1's to x_2. Then we distribute the remaining $32 - (5+5) = 22$ 1's, so there are

$$\binom{22+3-1}{3-1} = \binom{24}{2}$$

solutions with $x_1, x_2 \geq 5,\ x_3 \geq 0$. But we over counted, and we need to remove the solutions where $x_1, x_2 \geq 5$ and $x_3 \geq 21$. Giving 5 1's to x_1, 5 1's to x_2, and 21 1's to x_3, there are $32 - (5+5+21) = 1$ 1's left to distribute, which can be done in

$$\binom{1+3-1}{3-1} = \binom{3}{2} = 3$$

ways. Therefore, there are

$$\binom{24}{2} - 3$$

solutions with $x_1, x_2 \geq 5$, $0 \leq x_3 \leq 20$.

$\square$

Exercise 1.3.2. How many ways can 20 kanelbullar be distributed amongst 4 students if

(a) there are no restrictions?

(b) every student gets at least one?

(c) the fourth student cannot have more than 10?

Solution. (a) There are $\binom{20+4-1}{4-1} = \binom{23}{3}$ ways of distributing 20 kanelbullar amongst 4 students.

(b) Give each student 1 kanelbulle, and distributed the remaining 16 kanelbullar, which can be done in

$$\binom{16+4-1}{4-1} = \binom{19}{3}$$

ways.

(c) If we take the answer in (a), then we over counted the ways of distributing such that the fourth sutdent has more than 10. If we give the fourth student 11 kanelbullar, there are 9 kanelbullar remaining which can be distributed in

$$\binom{9+4-1}{4-1} = \binom{12}{3}$$

ways. Therefore, the number of ways of distributing 20 kanelbullar amongst 4 students such that the fourth student does not receive more than 10 is

$$\binom{19}{3} - \binom{12}{3}.$$

□

Exercise 1.3.3. How many ways can 12 apples and 7 muffins be distributed in 5 baskets if every basket must have at least 1 muffin?

Solution. There are $\binom{12+5-1}{5-1} = \binom{16}{4}$ ways of distributing 12 apples in 5 baskets. Giving each basket 1 muffin, there are $\binom{2+5-1}{5-1} = \binom{6}{4}$ ways of distributing the remaining 2 muffins. By the rule of product, there are a total of

$$\binom{16}{4}\binom{6}{4}$$

ways of distributing 12 apples and 7 muffins among 5 baskets such that each basket gets at least 1 muffin. □

Exercise 1.3.4. How many rearrangements of the letters of UPPSALA are there

(a) with no restrictions?

(b) that have no consecutive A's?

(c) that do not have U and S together?

Solution. (a) There are 2 P's and 2 A's, and 1 of the remaining letters. The total number of rearrangements is then

$$\binom{7}{2,2,1,1,1} = \frac{7!}{2!2!1!1!1!}.$$

(b) First, rearrange the letters UPPSL. There are

$$\binom{5}{2,1,1,1} = \frac{5!}{2!1!1!1!}$$

ways of doing this. For each rearrangements, there are 6 places between letters where A's can be placed, keeping them seperated, for example

$$\_P\_U\_L\_P\_S\_$$

There are $\binom{6}{2}$ ways of choosing 2 positions to place the A's. By the rule of product, there are

$$\binom{5}{2,1,1,1}\binom{6}{2}$$

rearrangements of UPPSALA with no consecutive A's.

(c) From (a) we know the number of rearrangements of UPPSALA, now we remove the ones where U and S are together. We introduce 2 letters, say $R := US$ and $T := SU$. There are $\binom{6}{2,2,1,1}$ rearrangements of PPLAAR. After replacing R with US, there are $\binom{6}{2,2,1,1}$ rearrangements of UPPSALA where US appears. Similarly, there are $\binom{6}{2,2,1,1}$ rearrangements of PPLAAT and $\binom{6}{2,2,1,1}$ rearrangements of Uppsala where SU appears. After removing these rearrangements, there are

$$\binom{7}{2,2,1,1,1} - 2\binom{6}{2,2,1,1}$$

rearrangements of UPPSALA where U and S are not together.

□

Exercise 1.3.5. How many Up/Right paths are there from $(0,0)$ to $(8,10)$

(a) with no restrictions?

(b) that go through $(4,7)$?

(c) that always take an even number of steps to the right? (for example, $RRURRRRUURR\ldots$ is allowed while $RRURUURRR\ldots$ is not.)

Solution. (a) This is the same as the number of rearrangments of 8 R's and 10 U's, of which there are

$$\binom{8+10}{8} = \binom{18}{8}.$$

(b) We count the paths from (0,0) to (4,7), which is given by

$$\binom{4+7}{4} = \binom{11}{4}.$$

Then we count the paths from (4,7) to (8,10), which will require going to the right 4 times and up 3 times. So there are

$$\binom{4+3}{4} = \binom{7}{4}$$

paths from (4,7) to (8,10). Each path from (0,0) to (8,10) that passes through (4,7) consists of first taking a path from (0,0) to (4,7), and then a path from (4,7) to (8,10), which can be done in

$$\binom{11}{4}\binom{7}{4}$$

ways.

(c) We can replace every pair RR with a new symbol, say $\mathcal{R}$. So instead of looking at rearrangements of RRRRRRRRUUUUUUUUUU, we look at rearrangements of $\mathcal{RRRR}$UUUUUUUUUU, which will guarantee that there are always an even number of steps to the right taken. There are

$$\binom{4+10}{4} = \binom{14}{4}.$$

such rearrangements.

□

Exercise 1.3.6. Suppose you are trying to find n people to volunteer to clean-up a park. You carry with you a sign-up sheet and n pens in a bag, in case all are used at once. Because you like math, you keep track of the number of pens in the bag in a sequence. For example, if $n = 3$ and all are used at once, then the sequence would go $3, 2, 1, 0, 1, 2, 3$ as they pick a pen one-by-one and return it one-by-one. If 2 people came at first, and later in the day a third person, the sequence would go $3, 2, 1, 2, 3, 2, 3$.

(a) How many such sequences are there if $n = 3$?

(b) How many such sequences are there if $n = 4$?

(c) How many such sequences are there for any positive number n? (HINT: for every sequence, place a $\nearrow$ between two numbers if the sequence increases, and a $\searrow$ if it decreases, for example

$$3 \searrow 2 \searrow 1 \nearrow 2 \nearrow 3 \searrow 2 \nearrow 3.$$

What can you say about the sequence of arrows?)

(a) 5 sequences:

3,2,1,0,1,2,3	3,2,3,2,1,2,1
3,2,1,2,1,2,3	3,2,3,2,3,2,3
3,2,1,2,3,2,3	

(b) 14 sequences:

4,3,2,1,0,1,2,3,4	4,3,2,3,4,3,2,3,4
4,3,2,1,2,1,2,3,4	4,3,2,3,4,3,4,3,4
4,3,2,1,2,3,2,3,4	4,3,4,3,2,1,2,3,4
4,3,2,1,2,3,4,3,4	4,3,4,3,2,3,2,3,4
4,3,2,3,2,1,2,3,4	4,3,4,3,2,3,4,3,4
4,3,2,3,2,3,2,3,4	4,3,4,3,4,3,2,3,4
4,3,2,3,2,3,4,3,4	4,3,4,3,4,3,4,3,4

(c) Every time someone takes a pen, the sequence goes down (so a $\searrow$ is placed between two numbers), and when that person returns the pen the sequence goes up (so a $\nearrow$ is placed). Of course, there cannot be more than n pens at any time, so there cannot be more $\nearrow$'s placed than $\searrow$'s placed at any time. Letting D denote $\searrow$ and U denote $\nearrow$, our desired sequences will have n D's and n U's in between the numbers such that there are never more U's than D's. So the number of such rearrangements of n D's and n U's is the same as the number of lattice paths from (0,0) to (n, n) that never go below the line $x = y$, which is given by the Catalan number

$$C(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Three Principles

1.1 Review of Principle of Mathematical Induction + Principle of Inclusion/Exclusion

Textbook readings

- From Keller + Trotter: Sections 3.1, 3.2, 3.6, 3.8, 3.9.
- From Keller + Trotter: Sections 7.1, 7.2.

Notation, Definitions, and Theorems

- **Well-ordering Principle:** Every non-empty set of positive integers has a minimal element.

Theorem 1.1.1 (Principle of Mathematical Induction). *Let $S(n)$ be an open statement involving the positive integer n. If*
Base Case: *$S(1)$ is true, and*
Inductive Step: *for all $k \geq 1$, if $S(k)$ is true then so is $S(k+1)$,*
then $S(n)$ is true for all $n \geq 1$.

Theorem 1.1.2 (Strong Induction). *Let $S(n)$ be an open statement involving the positive integer n. Let $1 \leq n_0 \leq n_1$. If*
Base Cases: *$S(n_0), S(n_0+1), \ldots, S(n_1-1), S(n_1)$ are true, and*
Inductive Step: *for all $k \geq n_1$, if $S(n_0), S(n_0+1), \ldots, S(k-1), S(k)$ are true then so is $S(k+1)$,*
then $S(n)$ is true for all $n \geq 1$.

Notation, Definitions, and Theorems

Theorem 1.1.3 (Principle of Inclusion/Exclusion). *Let X be a set, and let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a family of properties. For $S \subset \{1, 2, \ldots, m\}$, let $N(S)$ be the number of elements of X which satisfy (at least) P_i for all $i \in S$ (and $N(\emptyset) = |X|$). The number of elements of X that satisfy none of the properties in $\mathcal{P}$ is given by*

$$\sum_{S \subseteq \{1,2,\ldots,m\}} (-1)^{|S|} N(S).$$

Exercises

Suggested exercises from textbooks

- From Keller + Trotter: Section 3.11, exercises 9–13, 19.
- From Keller + Trotter: Section 7.7, exercises 1,2.

Exercise 1.1.1. Use mathematical induction and Pascal's identity to prove the hockey stick identity: for all nonnegative integers $0 \le r < n$,

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

Exercise 1.1.2. A local bakery sells kanelbullar in packages of 4 or 5. Use mathematical induction to prove that any number of kanelbullar above 11 can be ordered in packages of 4 or 5.

Exercise 1.1.3. At a large family barbecue, there are 75 people. All 75 people have a hotdog. On top of the hotdog, 45 people have potato salad, 45 have corn, 44 have coleslaw, 25 have potato salad and corn, 28 have potato salad and coleslaw, 26 have coleslaw and corn, and 15 have all of potato salad, corn, and coleslaw. How many people only ate hotdogs?

Exercise 1.1.4. The first few numbers in the Fibonacci sequence are $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. More formally, the sequence is defined recursively by $f_1 = 1, f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \ge 2$. Let r be the positive root of the quadratic equation $r^2 - r - 1 = 0$, so

$$r = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

Prove by mathematical induction that for all $n \ge 2$, $f_n \ge r^{n-2}$.

Remark 1.1.4. *This value $(1+\sqrt{5})/2$ is called the* **golden ratio**, *and often denoted by φ. It is known that*

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi.$$

Exercise 1.1.5. In this exercise, we will determine the maximum number of regions formed by n intersecting circles.

(a) What is the maximum number of distinct regions (including the outside region) formed by 2 intersecting circles? by 3 intersecting circles?

(b) Convince yourself that any 2 circles can intersect in at most 2 points. Use this fact to prove that a new circle can add at most $2n$ new regions to the number of regions formed by n other intersecting circles.

(c) Let $r(n)$ be the maximum number of regions formed by n intersecting circles. Use part (b) to prove that $r(n+1) = r(n) + 2n$ for all $n \ge 1$.

(d) Use part (c) and mathematical induction to prove that the maximum number of regions formed by n intersecting circles is $n^2 - n + 2$.

Remark 1.1.5. *A Venn diagram is used to represent all possibilities of elements belonging to n different sets. However, Venn diagrams are never (or rarely) used to represent more than 3 different sets (i.e., we never use 4 intersecting circles). That's because, for 4 sets, there are $2^4 = 16$ possibilities for which sets an element may belong to, but 4 intersecting circles form at most $4^2 - 4 + 2 = 14$ regions.*

1.2 Principle of Inclusion/Exclusion

Textbook readings

- From Keller + Trotter: Sections 7.1–7.4

Notation, Definitions, and Theorems

Theorem 1.2.1 (Principle of Inclusion/Exclusion)**.** *Let X be a set, and let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a family of properties. For $S \subset [m]$, let $N(S)$ be the number of elements of X which satisfy P_i for all $i \in S$ (and $N(\emptyset) = |X|$). The number of elements of X that satisfy none of the properties in $\mathcal{P}$ is given by*

$$\sum_{S \subseteq [m]} (-1)^{|S|} N(S).$$

- For $n \geq m \geq 1$, the Stirling number of the second kind is given by

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

- For $n \geq m$, the number of surjections from $[n]$ to $[m]$ is given by

$$S(n,m) = m! \left\{ {n \atop m} \right\} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

- The number d_n of derangements of $[n]$ satisfies

$$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!.$$

Exercises

Suggested exercises from textbooks

- From Keller + Trotter: Section 7.7, exercises 4,5, 7–10, 14–16, 18–20, 22.

Exercise 1.2.1. How many positive integers between 1 and 1000 are not divisible by 2,3, or 5?

Exercise 1.2.2. A local donut shop has 4 types of donuts. Currently, there are 3 chocolate donuts, 4 vanilla donuts, 13 strawberry jelly donuts, and 2 crullers. How many different ways can you choose a dozen donuts?

Exercise 1.2.3. In how many ways can the letters in UPPSALA be rearranged so that there are no occurrences of LAP, UP, SAP, or PAL?

Exercise 1.2.4. A retiring Mathematics professor has 7 textbooks she wants to hand out to her last 4 graduate students. In how many ways can she distribute her textbooks so that every student gets at least 1?

Exercise 1.2.5. There are 8 students sitting in a combinatorics class that only has 8 available seating places. During break, they leave to get coffee, and return to sit in different places. In how many ways can the students sit down after the break so that no student is seated where they were before the break.

1.3 Pigeonhole Principle

Textbook readings

- From Keller + Trotter: Section 4.1

Notation, Definitions, and Theorems

- For $n \geq 1$, we denote $[n] = \{1, 2, 3, \ldots, n\}$.
- **Pigeonhole Principle:** If m object occupy n places and $m > n$, then at least one place has two or more objects.
- **Generalized Pigeonhole Principle:** If m objects occupy n places and $m > kn + 1$, then at least once places has $k + 1$ or more objects.

Exercises

Suggested exercises from textbooks

- From Keller + Trotter: Section 4.6, exercises 2,3

Exercise 1.3.1. All of your socks are either black, white, or grey. How many socks do you need to pull from the dryer to guarantee that you have at least one pair of socks with matching colours?

Exercise 1.3.2. Prove that no matter how 5 points are placed on a sphere, there is a hemisphere that contains at least 4 of the points.

Exercise 1.3.3. Prove that no matter 19 integers are selected from the set $[35]$, two of the integers selected will sum to 36.

Exercise 1.3.4. Prove that no matter how 151 integers are chosen from the set $[300]$, there are two integers m and n so that $m|n$.

Exercise 1.3.5. Elin is an engineering student who drinks a lot of coffee at Café Ångström. During the month of November, she drank at least one cup of coffee a day, but drank at most 45 coffees altogether. Prove that there is a span of consecutive days during which she drank exactly 14 coffees.

Exercise 1.3.6. Your neighbour is having a yard sale, with everything priced between 1kr and 100kr. Show that for no matter which 10 selected objects, two nonempty piles of objects can be made from the selected objects such that the price of the items in each pile sum to the same number.

Exercise 1.3.7. Consider a board of 8 square by 2 squares (so there are 16 squares in total). Suppose we draw coloured circles at each of the 27 corners of squares, each circle is either red or blue. Prove that there is some rectangle on the board with all 4 corners having the same colour.

Thursday, February 4th

Review of Principle of Mathematical Induction + Principle of Inclusion/Exclusion

Principle of Mathematical Induction

The well-ordering principle: Every non-empty set of positive integers has a least element. $\{2, 3, 5, 7, 11, 13, \ldots\}$

smallest

Theorem (Principle of Mathematical Induction, PMI)

Let $S(n)$ be an open statement involving the positive integer n. If

we don't know if it's true or false....

Base Case: $S(1)$ is true, and

Inductive Step: for all $k \geq 1$, if $S(k)$ is true then so is $S(k+1)$

then $S(n)$ is true for all $n \geq 1$.

Proof: Suppose $S(n)$ satisfies the base case and the Inductive Step. Let $F = \{k \geq 1 : S(k) \text{ is false}\}$, the set of positive integers where $S(k)$ fails. If F is empty, then we're done. Otherwise, by the well-ordering principle, F has a least element m. Since the Base Case holds, $1 \notin F$, so $m \neq 1$. Since m is the least element, then $m-1 \notin F$, so $S(m-1)$ is true. But by the Inductive Step, so is $S(m)$ true and so $m \notin F$. This contradicts F having a least element, so F must be empty. $\square$

Example: Let $S(n)$ be the statement $\sum_{i=0}^{n-1} 2^i = 2^n - 1$. Prove that $S(n)$ is true for all $n \geq 1$.

Base Case: $n=1$. Then $\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2^1 - 1$, so $S(1)$ is true.

Inductive Step: Let $k \geq 1$, and assume $S(k)$ is true. This is called the Induction Hypothesis (IH). Then

$$\sum_{i=0}^{k} 2^i = \sum_{i=0}^{k-1} 2^i + 2^k \underset{\substack{\text{by } S(k) \text{ or} \\ \text{by I.H.}}}{=} 2^k - 1 + 2^k = 2(2^k) - 1 = 2^{k+1} - 1$$

Therefore $S(k+1)$ holds if we assume $S(k)$.

(By PMI, $S(n)$ is true for all $n \geq 1$.)

Important parts.

Theorem (Strong Induction): Let $S(n)$ be an open statement involving the positive integer n. Let $1 \leq n_0 \leq n_1$. If

Base Cases: $S(n_0), S(n_0+1), \dots, S(n_1-1), S(n_1)$ are true, and

Inductive Step: for all $k \geq n_1$, if $S(n_0), S(n_0+1), \dots, S(k-1), S(k)$ are true, then so is $S(k+1)$,

then $S(n)$ is true for all $n \geq n_0$.

Proof: Assume the Base Cases and the Inductive Step above hold. Let $P(n)$ be the statement "$S(n_0), S(n_0+1), \dots, S(n_1+n-1)$ are true".

Base case: $n=1$, $P(1)$ is true by the Base Cases above.

Inductive Step: Let $k \geq 1$, and assume $P(k)$ is true. Then $S(n_0), \dots, S(n_1+k-1)$ are true. By the inductive step above, this implies $S(n_1+k)$ is true. Therefore, $S(n_0), \dots, S(n_1+k-1), S(n_1+k)$ are true, so $P(k+1)$ is true.

So by PMI, $P(n)$ is true for all $n \geq 1$, and $P(n-n_1+1)$ implies $S(n)$, so $S(n)$ is true for all $n \geq 1$ $\square$

Example: You can buy mozzarella sticks in bags of 3 or 5 at Max. Show that for any $n \geq 8$, you can buy exactly n mozzarella sticks.

Let $S(n)$ be the statement "$n = 3a + 5b$ for some nonnegative integers a and b". If $S(n)$ is true, then you can buy exactly n mozzarella sticks.

Base Cases: $n = 8$, $n = 3(1) + 5(1)$

$n = 9$, $n = 3(3) + 5(0)$

$n = 10$, $n = 3(0) + 5(2)$

So $S(8)$, $S(9)$, and $S(10)$ are true.

Inductive Step: Let $k \geq 10$, and assume $S(8), S(9), \dots, S(k)$ are true. Then $k-2 \geq 8$, so $S(k-2)$ is true, so by I.H., $k-2 = 3a+5b$ for nonnegative a, b. Then

$$k+1 = (k-2)+3 \underset{\text{by I.H.}}{=} (3a+5b)+3 = 3(a+1)+5b,$$

so there exists nonnegative integers $a+1, b$ such that $k+1 = 3(a+1)+5b$, so $S(k+1)$ is true.

By Strong Induction, $S(n)$ is true for all $n \geq 8$.

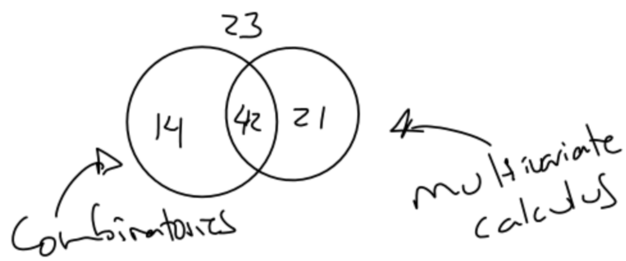
$\square$

Principle of Inclusion/Exclusion

Example: Out of 100 students 56 are registered in the combinatorics course, 63 in multivariate calculus while 42 are registered for both. How many students are taking neither course?

Start by subtracting 56 and 63 from 100 to get $100-63-56=-19$, which makes no sense. But that's because we subtracted the 42 students taking both courses twice, so we must add them back in. So the number of students taking neither class is

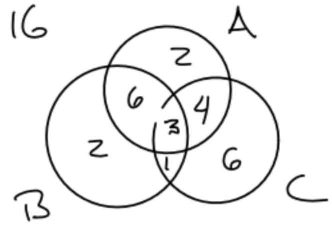
$$100-63-56+42=23.$$



Example: Let X be a set with $|X|=40$ elements. Let $A,B,C\subseteq X$ with $|A|=15$, $|B|=12$, $|C|=14$, $|A\cap B|=9$, $|A\cap C|=7$, $|B\cap C|=4$, $|A\cap B\cap C|=3$. How many elements are in $S\setminus(A\cup B\cup C)$, the set of elements in none of $A,B,$ or C?

Again start by subtracting 15, 12, 14 from 40. But we subtracted the elements in $A\cap B$, $A\cap C$, $B\cap C$ twice, so we add 9, 7, 4 back. But the elements of $A\cap B\cap C$ have been removed 3 times (as elements of A,B,C), and added back in 3 times (as elements of $A\cap B, A\cap C, B\cap C$). So we need to remove those 3 elements. So,

$$|S\setminus(A\cup B\cup C)|=40-15-12-14+9+7+4-3=16$$



Notation: Let X be a set,

- Let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a family of properties
  for example, – P_i is the property "element of set A_i"
  – P_i is the property "≥ 3 and < 7"
- For $S \subseteq \{1, 2, \ldots, m\}$, let $N(S)$ be the number of elements of X that satisfy AT LEAST the properties $\{P_i : i \in S\}$
  - For $S = \emptyset \subseteq \{1, 2, \ldots, m\}$, $N(\emptyset)$ is the number of elements of X satisfying at least no properties, so $N(\emptyset) = |X|$
  - Clearly, $N(\{1\}) \geq N(\{1,2\})$ for example

For example, look at example above, and let

$P_1 :=$ "belongs to A"
$P_2 :=$ "belongs to B"
$P_3 :=$ "belongs to C".

Then $N(\emptyset) = 40$, $N(\{1\}) = 15$, $N(\{2\}) = 12$, $N(\{3\}) = 14$
$N(\{1,2\}) = 9$, $N(\{1,3\}) = 7$, $N(\{2,3\}) = 4$, $N(\{1,2,3\}) = 3$.

And we saw that the number of elements satisfying none of P_1, P_2, P_3 was given by

$$N(\emptyset) \underbrace{- N(\{1\}) - N(\{2\}) - N(\{3\})}_{(-1)^1} \underbrace{+ N(\{1,2\}) + N(\{1,3\}) + N(\{2,3\})}_{(-1)^2} \underbrace{- N(\{1,2,3\})}_{(-1)^3}$$

Theorem (Principle of Inclusion\Exclusion): Let X be a set and let $\mathcal{P}=\{P_1,P_2,\dots,P_m\}$ be a family of properties. The number of elements of X that satisfy none of the properties in $\mathcal{P}$ is given by

$$\sum_{S\subseteq\{1,2,\dots,m\}} (-1)^{|S|}N(S).$$

sum over all possible subsets $S\subseteq\{1,2,\dots,m\}$.

There is a proof by induction in the textbook, but here is a combinatorial proof.

Proof: Let N^* be the number of elements of X satisfying none of $\mathcal{P}=\{P_1,\dots,P_m\}$. So the claim is that

LHS: Left hand side

RHS: right hand side

$$N^*=\sum_{S\subseteq\{1,2,\dots,m\}} (-1)^{|S|}N(S) \qquad (*)$$

Take any $x\in X$.

If x satisfies none of $P_1,\dots,P_m$, then it is counted once on the LHS of $(*)$. It is also counted in $N(\emptyset)=|X|$, and in no other $N(S)$ for $S\neq\emptyset$. So x is also counted once on the RHS of $(*)$.

If x satisfies exactly r properties for $1\le r\le m$, then it contributes 0 to the LHS of $(*)$. As for the RHS,

There is 1 empty set $\emptyset\subseteq\{1,2,\dots,m\}$ where x is counted in $N(\emptyset)$

There are r sets S with $|S|=1$ where x is counted in $N(S)$

There are $\binom{r}{2}$ sets S with $|S|=2$ where x is counted in $N(S)$

There are $\binom{r}{3}$ sets S with $|S|=3$ where x is counted in $N(S)$

$\vdots$

There are $\binom{r}{r}=1$ set S with $|S|=r$ where x is counted in $N(S)$

By the Binomial Theorem, x contributes

$$1-r+\binom{r}{2}-\binom{r}{3}+\cdots+(-1)^r\binom{r}{r}=\sum_{k=0}^{r}\binom{r}{k}(-1)^k=(1+(-1))^r=0$$ to the RHS

of $(*)$. So both sides of $(*)$ count the same elements, so the equality holds $\square$

Monday, February 8th

Principle of Inclusion/Exclusion

Recall from last time:

Notation: - Let X be a set,
- Let $\mathcal{P}=\{P_1,P_2,\dots,P_m\}$ be a family of properties
- For $S\subseteq\{1,2,\dots,m\}$, let $N(S)$ be the number of elements of X that satisfy AT LEAST the properties $\{P_i : i\in S\}$

Theorem (Principle of Inclusion\Exclusion): Let X be a set and let $\mathcal{P}=\{P_1,P_2,\dots,P_m\}$ be a family of properties. The number of elements of X that satisfy none of the properties in $\mathcal{P}$ is given by

$$\sum_{S\subseteq\{1,2,\dots,m\}} (-1)^{|S|}N(S).$$

Example: There are how many integer solutions to

$$x_1+x_2+x_3=20, \quad \text{with } 0\le x_1\le 8, \quad 0\le x_2\le 10, \quad 0\le x_3\le 12 \;?$$

Let $X=\{(x_1,x_2,x_3) : x_1+x_2+x_3=20,\ x_1,x_2,x_3\ge 0\}$. We know that there are $\binom{20+3-1}{3-1}=\binom{22}{2}$ integer solutions with $x_1,x_2,x_3\ge 0$, so $|X|=\binom{22}{2}$.

Let P_1 be the property $x_1\ge 9$, P_2 be the property $x_2\ge 11$, and P_3 be the property $x_3\ge 13$.

$N(\emptyset)=|X|=\binom{22}{2}$

Giving 9 1's to x_1, there are $\binom{11+3-1}{3-1}=\binom{13}{2}$ solutions where $x_1\ge 9$, $x_2,x_3\ge 0$

So $N(\{1\})=\binom{13}{2}$

Giving 11 1's to x_2, there are $\binom{9+3-1}{3-1}=\binom{11}{2}$ solutions where $x_2\ge 11$, $x_1,x_3\ge 0$

So $N(\{2\})=\binom{11}{2}$

Giving 13 1's to x_3, there are $\binom{7+3-1}{3-1}=\binom{9}{2}$ solutions where $x_3\geq 13$, $x_1,x_2\geq 0$

Giving 9 1's to x_1, and 11 1's to x_2, there are $\binom{0+3-1}{3-1}=\binom{2}{2}=1$ solution where $x_1\geq 9$, $x_2\geq 11$, $x_3\geq 0$, so $N(\{1,2\})=1$

There are 0 solutions where $x_1\geq 9, x_2\geq 0, x_3\geq 13$, so $N(\{1,3\})=0$

——〃—— $x_1\geq 0, x_2\geq 11, x_3\geq 13$, so $N(\{2,3\})=0$

——〃—— $x_1\geq 9, x_2\geq 11, x_3\geq 13$, so $N(\{1,2,3\})=0$.

So by the Principle of Inclusion/Exclusion, the number of solutions to $x_1+x_2+x_3=0$, $0\leq x_1\leq 8$, $0\leq x_2\leq 10$, $0\leq x_3\leq 12$ is given by

$$N(\emptyset)-N(\{1\})-N(\{2\})-N(\{3\})+N(\{1,2\})+N(\{1,3\})+N(\{2,3\})-N(\{1,2,3\})$$
$$=\binom{22}{2}-\binom{13}{2}-\binom{11}{2}-\binom{9}{2}+1+0+0-0$$
$$=\binom{22}{2}-\binom{13}{2}-\binom{11}{2}-\binom{9}{2}+1.$$

<u>Example</u>: How many integers between 1 and 100 are not divisible by 2, 3, or 5?

Let $X=\{1,2,\ldots,100\}$.
Let P_1 be the property "divisible by 2"
Let P_2 be the property "divisible by 3"
Let P_3 be the property "divisible by 5".

$N(\emptyset)=|X|=100$

There are $\lfloor 100/2\rfloor=50$ even numbers in X, so $N(\{1\})=50$

There are $\lfloor 100/3\rfloor=33$ multiples of 3 in X, so $N(\{2\})=33$

There are $\lfloor 100/5\rfloor=20$ multiples of 5 in X, so $N(\{3\})=20$

There are $\lfloor 100/6\rfloor=16$ multiples of 6 in X, so $N(\{1,2\})=16$ # in X divisible by 2 and by 3.

There are $\lfloor 100/10\rfloor=10$ multiples of 10 in X, so $N(\{1,3\})=10$

There are $\lfloor 100/15\rfloor=6$ multiples of 15 in X, so $N(\{2,3\})=6$

There are $\lfloor 100/30\rfloor=3$ multiples of 30 in X, so $N(\{1,2,3\})=3$

So by the Principle of Inclusion/Exclusion, the number of integers between 1 and 100 that are not divisible by 2, 3, or 5 is given by

$$100-50-33-20+16+10+6-3=26.$$

Enumerating Surjections

- Let A, B be two sets, and $f: A \to B$ a function.
- $f(A) = \{b \in B : b = f(a) \text{ for some } a \in A\}$, the image of A under f.
- f is a <u>surjection</u> if $f(A) = B$
- If A, B are finite sets and f is a surjection, then $|A| \geq |B|$.

We want to count the total number of surjections from A to B.

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ with $n \geq m$.

Say that a function $f: A \to B$ satisfies the property P_i if $b_i \notin f(A)$.

<u>Lemma</u>: Let $S \subseteq \{1, 2, \ldots, m\}$ with $|S| = k$. The number of functions satisfying P_i for every $i \in S$ is $(m-k)^n$.

<u>Proof</u>: Let $C = \{b_i : i \in S\}$, then $|C| = k$. We can think of functions from A to B as strings of length n taking elements from B where f maps a_j to the element in position j

<u>ex</u>: $b_1 b_3 b_1 b_4 b_3 \iff f(a_1) = b_1, f(a_2) = b_3, f(a_3) = b_1, f(a_4) = b_4, f(a_5) = b_3$

So we can think of functions satisfying P_i for all $i \in S$ as strings of length n taking elements from $B \setminus C$. There are a total of $|B| - |C| = m-k$ elements in $B \setminus C$, so there are

$$\underbrace{(m-k)(m-k)(m-k)\cdots(m-k)}_{n \text{ times}} = (m-k)^n \text{ such strings.} \quad \square$$

<u>Definition</u>: For $n \geq m \geq 1$, the <u>Stirling number of the second kind</u> is given by

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

Theorem: Let A, B be finite sets with $|A| = n$, $|B| = m$ and $n \geq m$. The number of surjections from A to B is given by

$$S(n,m) = m!\left\{ {n \atop m} \right\} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

Proof: Keeping the notation from above, a surjection is a function that fails all of $P_1, \dots, P_m$. For each k, there are $\binom{m}{k}$ subsets $S \subseteq \{1, 2, \dots, m\}$ with $|S| = k$, and for each S, by the previous lemma, there are $N(S) = (m-k)^n$ functions satisfying P_i for all $i \in S$. Therefore, by the principle of Inclusion/Exclusion, there are

$$S(n,m) = \sum_{S \subseteq \{1,2,\dots,m\}} (-1)^{|S|} N(S) = \sum_{k=0}^{m} (-1)^k \underbrace{\binom{m}{k}}_{\text{\# of subsets of size } k} \overbrace{(m-k)^n}^{N(S) \text{ when } |S| = k}$$

surjections from A to B. $\square$

Example: Grandma Agnes knitted 5 distinct blankets. In how many ways could she give the blankets to her 3 grandchildren such that every grandchild gets at least 1?

This is the number of surjections from the set of blankets to the set of grandchildren (It's a surjection since every grandchild receives at least 1 blanket), which is

$$S(\underset{n}{5},\underset{m}{3}) = \sum_{k=0}^{3} (-1)^k \binom{3}{k} (3-k)^5 = \binom{3}{0}3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5 - 0$$

$$= 1 \cdot 243 - 3 \cdot 32 + 3 \cdot 1 - 0$$

$$= 150.$$

Derangements

Recall: We defined a permutation as a string $s:\{1,2,\dots,k\}\to X$, denoted $s := x_1x_2\dots x_k$, where all x_i are different.

- It's also common to reserve permutations for strings $\sigma:\{1,2,\dots,n\}\to\{1,2,\dots,n\}$.

- A <u>derangement</u> is a permutation σ such that $\sigma(i)\neq i$ for all $1\le i\le n$.

For example, $\sigma := 4312$ is a derangement since $\sigma(1)=4$, $\sigma(2)=3$, $\sigma(3)=1$, $\sigma(4)=2$

$\sigma := 4213$ is <u>not</u> a derangement since $\sigma(2)=2$

Theorem: The number of derangements d_n of $\{1,2,\dots,n\}$ is given by

$$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)!$$

Proof: Let P_i be the property that $\sigma(i)=i$. There are a total of $n!$ permutations, so $N(\emptyset)=n!=(-1)^0\binom{n}{0}(n-0)!$

For $S\subseteq\{1,2,\dots,n\}$, the number of permutations satisfying P_i for $i\in S$ is given by $N(S)=(n-|S|)!$: Let's build σ satisfying P_i for $i\in S$. Fix $\sigma(i)=i$ for $i\in S$, and permute the $n-|S|$ remaing integers into the remaing positions of σ. There are $(n-|S|)!$ ways of doing this.

For each k, there are $\binom{n}{k}$ subsets $S\subseteq\{1,2,\dots,n\}$ with $|S|=k$, so by the Principle of Inclusion/Exclusion, the number of permutations satisfying none of $P_1,P_2,\dots,P_n$ (# of derangements) is

$$d_n = \sum_{S\subseteq\{1,2,\dots,n\}} (-1)^{|S|}N(S) = \sum_{k=0}^{n} (-1)^k \underbrace{\binom{n}{k}}_{\#\text{ of subsets } S \text{ of size } k} \overbrace{(n-k)!}^{(n-|S|)!}.$$

$\square$

Example: 5 friends want to read 5 different books. They each buy one of the books. Once they have read their book, they want to make an exchange such that no one gets their book back. In how many ways can this be done?

Label the friends and their purchased book by 1, 2, 3, 4, 5. Then any derangement σ of $\{1,2,3,4,5\}$ will correspond to an assignment $\sigma(i)=j$, where person i receives book j, and no one gets their book back. There are

ex: $\begin{array}{ccccccc} 5 & 3 & 1 & 2 & 4 & \leftarrow & \text{books} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\ 1 & 2 & 3 & 4 & 5 & \leftarrow & \text{people} \end{array}$

$$d_5 = \sum_{k=0}^{5} (-1)^k \binom{5}{k}(5-k)!$$

$$= \cancel{\binom{5}{0}5!} - \cancel{\binom{5}{1}4!} + \binom{5}{2}3! - \binom{5}{3}2! + \binom{5}{4}1! - \binom{5}{5}0!$$

$$= 1\cdot 120 - 5\cdot 24 + 10\cdot 6 - 10\cdot 2 + 5\cdot 1 - 1\cdot 1$$

$$= 44$$

Ex: $\displaystyle\sum_{S\subseteq\{1,2,3\}} (-1)^{|S|} N(S) = \sum_{k=0}^{3} (-1)^k \binom{3}{k}(3-k)^n$ (from surjection proof)

$\binom{3}{0}$ sets

↳ $(-1)^0 N(\emptyset) + (-1)^1 N(\{1\}) + (-1)^1 N(\{2\}) + (-1)^1 N(\{3\})$ ← $\binom{3}{1}$ sets

$+ (-1)^2 N(\{1,2\}) + (-1)^2 N(\{1,3\}) + (-1)^2 N(\{2,3\})$ ← $\binom{3}{2}$ sets

$+ (-1)^3 N(\{1,2,3\})$ ← $\binom{3}{3}$ sets

$N(S) = (m-k)^n$ if $|S| = k$

$$= \sum_{k=0}^{3} (-1)^k \binom{3}{0}(3-k)^n.$$

Wednesday, February 10^{th}

Pigeonhole Principle

notation: $[n] := \{1, 2, \ldots, n\}$.

Pigeonhole Principle (PHP): If m objects (pigeons) occupy n places (pigeonholes) and $m > n$, then one place has at least 2 objects.

Example: If there are 13 students in the classroom, then at least 2 students have a birthday in the same month.

- Students are pigeons
- 12 months of the year are the pigeonholes.

Since $13 > 12$, one month has at least 2 students.

Generalized Pigeonhole Principle (GPHP): If m objects (pigeons) occupy n places (pigeonholes), and $m > kn$, then at least 1 place has at least $k+1$ objects.

Example: If there are 37 students in the classroom, then at least 4 have a birthday in the same month

- 37 students are the pigeons
- 12 months of the year are the pigeonholes

$m = 37$, $n = 12$, $k = 3$, $m = kn + 1$, so some pigeonhole (month) has at least $k + 1 = 4$ students.

Example: A jewelery booth at the town square sells rings with 4 gems placed in a row, each gem taking one of 3 colours. Show that if the store has 82 rings, then 2 rings have identical sequences of gems.

- 82 rings are the pigeons
- Each gem can be one of 3 colours, so there are $3 \times 3 \times 3 \times 3 = 81$ possible sequences of gems, so these are the 81 pigeonholes

By PHP, 2 rings have the same sequence of gems.

Example: For any $A \subseteq [200] = \{1, 2, \ldots, 200\}$ with $|A| = 101$, there exists $m, n \in A$ such that $n | m$. ← "n divides m" or "m is a multiple of n"

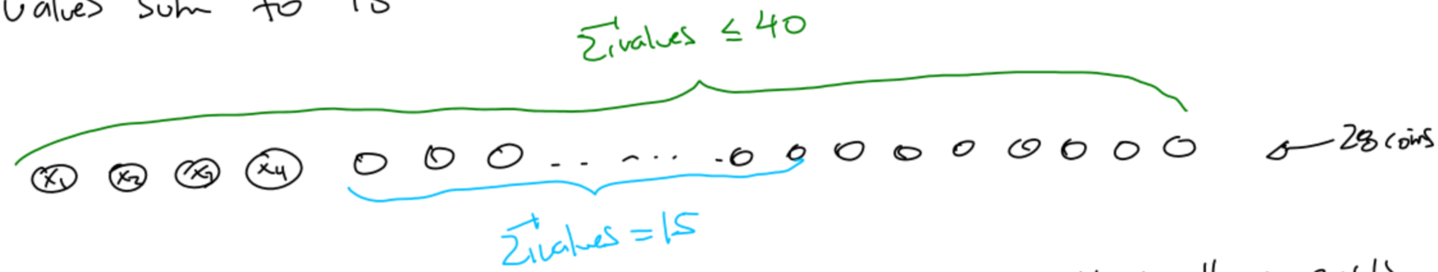
- Let the elements of A be the pigeons.
- For the pigeonholes, look at the 100 sets

$$\to \{1, 1\cdot 2, 1\cdot 2^2, 1\cdot 2^3, \ldots, 1\cdot 2^i, \ldots\}$$
$$\{3, 3\cdot 2, 3\cdot 2^2, 3\cdot 2^3, \ldots, 3\cdot 2^i, \ldots\}$$
$$\{5, 5\cdot 2, 5\cdot 2^2, 5\cdot 2^3, \ldots, 5\cdot 2^i, \ldots\}$$
$$\vdots$$
$$\{199, 199\cdot 2, 199\cdot 2^2, 199\cdot 2^3, \ldots, 199\cdot 2^i, \ldots\}$$

For all numbers $n \in [200]$, $n = 2^k \cdot g$ where g is an odd number, then n goes in the pigeonhole $\{g, g\cdot 2, g\cdot 2^2, \ldots, g\cdot 2^i, \ldots\}$

So all 101 pigeons are in the pigeonholes, So by PHP, 2 numbers $n = g\cdot 2^{k_1}$, $m = g\cdot 2^{k_2}$ are in the same pigeonhole, where $k_2 > k_1$. Then $n|m$ since $\frac{m}{n} = \frac{\cancel{g}\, 2^{k_2}}{\cancel{g}\cdot 2^{k_1}} = 2^{k_2 - k_1}$ is a whole integer

Example: 28 coins with values 1Kr, 2Kr, 5Kr or 10Kr, are placed in a row on a table, and the sum of the values of the coins does not exceed 40. There is a sequence of consecutive coins whose values sum to 15



Let x_i be the value of coin i, we want to show that there exists $x_{i+1}+x_{i+2}+\cdots+x_{j-1}+x_j=15$. Let $y_i = x_1+x_2+\cdots+x_i$, the sum of the values of the first i coins. Since each $x_i \geq 1$, and $y_{28} \leq 40$, then

$$1 \leq y_1 < y_2 < y_3 < \cdots < y_{28} \leq 40.$$

Also, we have

$$16 \leq y_1+15 < y_2+15 < y_3+15 < \cdots < y_{28}+15 \leq 55.$$

There are 56 numbers $y_1, y_2, y_3, \ldots, y_{28}, y_1+15, y_2+15, \ldots, y_{28}+15$, these are our pigeons. The pigeonholes are the 55 values $\{1, 2, \ldots, 55\}$ that these numbers can take. By the PHP, two of these numbers have the same value. Since $y_i < y_j$ and $y_i+15 < y_j+15$ if $i<j$, the two numbers that are equal are y_j and y_i+15 for some $i<j$. Then the values of the coins $x_{i+1}, x_{i+2}, \ldots, x_j$ sum to

$$x_{i+1}+x_{i+2}+\cdots+x_{j-1}+x_j = (x_1+x_2+\cdots+x_i+x_{i+1}+\cdots+x_j) - (x_1+x_2+\cdots+x_i)$$

$$= y_j - y_i = 15$$

Example: Take any subset $A \subseteq [9] = \{1, 2, \dots, 9\}$ with $|A| = 6$.
Then A contains two elements $x, y \in A$ such that $x + y = 10$

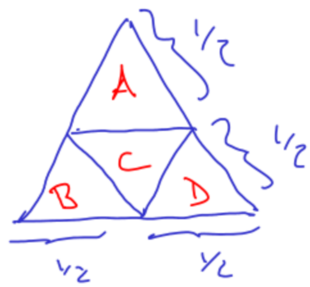
Here we let the 6 elements of A be the pigeons, and consider the 5 pigeonholes

$$\{1, 9\}, \ \{2, 8\}, \ \{3, 7\}, \ \{4, 6\}, \ \{5\}$$

By PHP, some pigeonhole contains two elements x and y.
$|\{5\}| = 1$ so the pigeonhole with 2 elements is one of the other sets, and so $x + y = 10$.

Example: Suppose 5 points are placed in a equilateral triangle with side lengths 1. Then there are two points whose distance apart is at most $1/2$.

Consider the 5 points to be the pigeons. Split the triangle into 4 smaller triangles A, B, C, D



So by PHP, one of $A, B, C,$ or D contains 2 points x, y and the maximum distance within a smaller triangle is $1/2$, so the distance between x and y is at most $1/2$.

Example: Let $S \subseteq [14]$ with $|S| = 6$. Then there are 2 subsets $A, B \subseteq S$ whose elements sum to the same value.

There are $2^6 = 64$ subsets of S. Any subset $A \subseteq S$ has a sum s_A satisfying $0 \leq s_A \leq 9 + 10 + 11 + 12 + 13 + 14 = 69$.
We want subsets to be the pigeons and possible sums to be the pigeonholes, but there are too many pigeonholes!
But we can't have $|A| = 6$, since if $|A| = 6$, then $A = S$, and $B \subsetneq S$ cannot have elements with the same sum as those of A.
Similarly, we can't have $|A| = 0$, since B will have at least 1 element, whose sum is clearly not equal to 0.
So consider only subsets $A, B \subseteq S$ with $1 \leq |A|, |B| \leq 5$. There are $2^6 - 2 = 62$ such subsets (all except $\emptyset$ and S). The possible sums s_A and s_B satisfy $1 \leq s_A, s_B \leq 10 + 11 + 12 + 13 + 14 = 60$. With 62 pigeons and 60 pigeonholes, 2 subsets $A, B \subseteq S$ have the same sum $s_A = s_B$.

Three Principles

1.1 Review of Principle of Mathematical Induction + Principle of Inclusion/Exclusion

Exercise 1.1.1. Use mathematical induction and Pascal's identity to prove the hockey stick identity: for all nonnegative integers $0 \leq r < n$,

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

Solution. We prove the hockey stick identity by fixing $r \geq 0$, and performing induction on n.
Base Case: $n = r$. Then

$$\sum_{k=r}^{n} \binom{k}{r} = \sum_{k=r}^{r} \binom{r}{r} = 1 = \binom{r+1}{r+1} = \binom{n+1}{r+1},$$

which proves the hockey stick identity when $n = r$.
Inductive Step: Let $m \geq r$, and assume that

$$\sum_{k=r}^{m} \binom{k}{r} = \binom{m+1}{r+1}. \tag{1.1}$$

Then

$$\sum_{k=r}^{m+1} \binom{k}{r} = \sum_{k=r}^{m} \binom{k}{r} + \binom{m+1}{r} \overset{\text{by 1.1}}{=} \binom{m+1}{r+1} + \binom{m+1}{r} \overset{\text{by Pascal's Identity}}{=} \binom{m+2}{r+1}.$$

Therefore, $\sum_{k=r}^{m+1} \binom{k}{r} = \binom{m+2}{r+1}$ follows from the induction hypothesis.

By the PMI, $\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$ for all $n \geq r$. Since r was arbitrary, this holds for all $r \geq 0$ as well. □

Exercise 1.1.2. A local bakery sells kanelbullar in packages of 4 or 5. Use mathematical induction to prove that any number of kanelbullar above 11 can be ordered in packages of 4 or 5.

Solution. Let $S(n)$ be the statement "$n = 4a + 5b$ for some nonnegative integers a and b".
Base Cases: $S(12), S(13), S(14)$ and $S(15)$ are true since

$$\begin{aligned} 12 &= 4(3) + 5(0), \\ 13 &= 4(2) + 5(1), \\ 14 &= 4(1) + 5(2), \\ 15 &= 4(0) + 5(3). \end{aligned}$$

Inductive Step: Let $k \geq 15$ and assume $S(12), S(13), \ldots, S(k)$ are all true. Since $S(k-3)$ is assumed to be true, then $k - 3 = 4a + 5b$ for nonnegative integers a, b. Then

$$k + 1 = (k-3) + 4 \overset{\text{by I.H.}}{=} 4a + 5b + 4 = 4(a+1) + 5b,$$

and since $a + 1, b$ are nonnegative integers, $S(k+1)$ is also true.

By Strong Induction, $S(n)$ is true for all $n \geq 12$. □

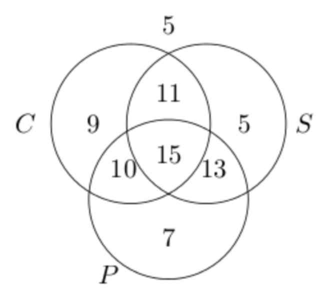
Exercise 1.1.3. At a large family barbecue, there are 75 people. All 75 people have a hotdog. On top of the hotdog, 45 people have potato salad, 45 have corn, 44 have coleslaw, 25 have potato salad and corn, 28 have potato salad and coleslaw, 26 have coleslaw and corn, and 15 have all of potato salad, corn, and coleslaw. How many people only ate hotdogs?

Solution. Let H be the set of people who had hotdogs, let P be the set of people who had potato salad, let C be the set of people who had corn, and let S be the set of people who had coleslaw. So $|H| = 75$ and

$$\begin{aligned} &|P| = 45, \quad |C| = 45, \quad |S| = 44, \\ &|P \cap C| = 25, \quad |P \cap S| = 28, \quad |C \cap S| = 26, \\ &|P \cap C \cap S| = 15. \end{aligned}$$

Then the number of people who ate only hotdogs is given by

$$|H \setminus (P \cup C \cup S)| = 75 - 45 - 45 - 44 + 25 + 28 + 26 - 15 = 5.$$



□

Exercise 1.1.4. The first few numbers in the Fibonacci sequence are $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. More formally, the sequence is defined recursively by $f_1 = 1, f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Let r be the positive root of the quadratic equation $r^2 - r - 1 = 0$, so

$$r = \frac{1 + \sqrt{5}}{2} \approx 1.618.$$

Prove by mathematical induction that for all $n \geq 2$, $f_n \geq r^{n-2}$.

Solution. We use strong induction. Let $S(n)$ be the statement $f_n \geq r^{n-2}$.
Base Cases: For $n = 2, 3$. Then $f_2 = 1 \geq 1 = r^0$ and $f_3 = 2 \geq r^1$, so $S(2)$ and $S(3)$ are true.
Inductive Step: Let $k \geq 3$, and assume $S(2), S(3), \ldots, S(k)$ are all true. Then

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ &\geq r^{k-2} + r^{k-3} && \text{by the Induction Hypothesis} \\ &= r^{k-3}(r+1) \\ &= r^{k-3}(r^2) && \text{since } r^2 - r - 1 = 0, \text{ so } r + 1 = r^2 \\ &= r^{k-1}. \end{aligned}$$

Therefore $S(k+1)$ follows from the induction hypothesis.

By Strong Induction, $S(n)$ is true for all $n \geq 2$. □

Exercise 1.1.5. In this exercise, we will determine the maximum number of regions formed by n intersecting circles.

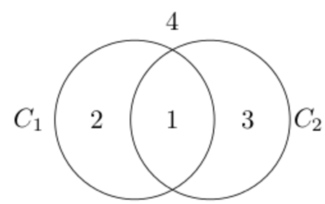
(a) What is the maximum number of distinct regions (including the outside region) formed by 2 intersecting circles? by 3 intersecting circles?

(b) Convince yourself that any 2 circles can intersect in at most 2 points. Use this fact to prove that one circle can intersect at most $2n$ of the regions formed by n other intersecting circles.

(c) Let $r(n)$ be the maximum number of regions formed by n intersecting circles. Use part (b) to prove that $r(n+1) = r(n) + 2n$ for all $n \geq 1$.

(d) Use part (c) and mathematical induction to prove that the maximum number of regions formed by n intersecting circles is $n^2 - n + 2$.

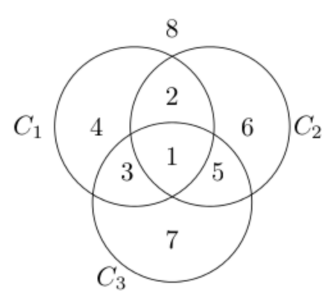
Solution. (a) For 2 circles C_1 and C_2, any region is either within C_1 or outside of it, and is either within C_2 or outside. This makes a maximum of 4 possible regions: 1: in C_1 and in C_2, 2: in C_1 and out of C_2, 3: out of C_1 and in C_2, and 4: out of C_1 and out of C_2. This is achieved by the following example:



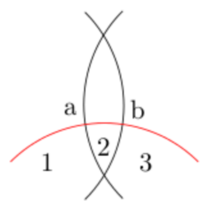
For three circles C_1, C_2 and C_3, there are a total of 8 possibilities,

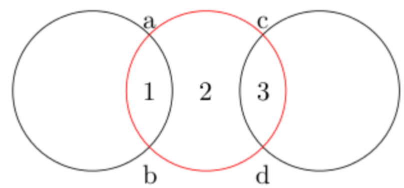
1. in C_1, in C_2, in C_3,
2. in C_1, in C_2, out of C_3,
3. in C_1, out of C_2, in C_3,
4. in C_1, out of C_2, out of C_3,
5. out of C_1, in C_2, in C_3,
6. out of C_1, in C_2, out of C_3,
7. out of C_1, out of C_2, in C_3,
8. out of C_1, out of C_2, out of C_3,

And this number of regions is achieved with the following example:



(b) Let C_{n+1} be a new circle introduced to n already intersecting circles. If C_{n+1} doesn't intersect a previous circle, then only 1 new region is formed, and clearly $1 \leq 2n$. Now suppose C_{n+1} does intersect previous circles. Let I be the number of points where C_{n+1} intersects another circle, and let R be new regions formed. We say a region is 'new' if it is contained inside C_{n+1} and touches the circle C_{n+1}. Since C_{n+1} intersects the previous circles at most twice, then $I \leq 2n$. Each new region touches at least 2 points where C_{n+1} intersects one of the previous circles, and each point where C_{n+1} intersects a previous circle is adjacent to at most 2 new regions.





For example, in the picture above to the left, the new region 2 touches the points a and b, while point a is adjacent to regions 1 and 2 and the point b is adjacent to regions 2 and 3. In the picture to the right, the new region 2 touches all four points a,b,c,d, while the new regions 1 and 3 both touch two points of intersection. Each of the points a,b,c,d touch 2 new regions.

So if we count two new regions for every point of intersection, then each new region was counted at least twice. So we can conlcude that

$$2R \leq 2I \leq 2(2n),$$

the last inequality coming from the fact that $I \leq 2n$. Therefore, we see that $R \leq 2n$.

(c) If $r(n)$ is the number of regions formed by n intersecting circles, then from above we see that $r(n+1) = r(n) + 2n$, since there are at most $2n$ new regions formed.

(d) Let $S(n)$ be the statement $r(n) = n^2 - n + 2$.

Base Cases: For $n = 1$, one circle creates two regions, so $r(1) = 2 = 1^2 - 1 + 2$, so $S(1)$ holds. The cases $n = 2$ and $n = 3$ were both covered in part (a), and we saw that

$$r(2) = 4 = 2^2 - 2 + 2$$

and

$$r(3) = 8 = 3^3 - 3 + 2.$$

Inductive Step: Let $k \geq 1$ and assume $S(k)$ is true, that is $r(k) = k^2 - k + 2$. Then by using part (c), we have

$$\begin{aligned} r(k+1) &= r(k) + 2k \\ &= k^2 + 2k + 2 + 2k && \text{By the induction hypothesis} \\ &= k^2 + 2k + 1 - k - 1 + 2 \\ &= (k+1)^2 - (k+1) + 2, \end{aligned}$$

so $S(k+1)$ is true from the induction hypothesis.

By the PMI, $S(n)$ is true for all $n \geq 1$.

$\square$

1.2 Principle of Inclusion/Exclusion

Exercise 1.2.1. How many positive integers between 1 and 1000 are not divisible by 2,3, or 5?

Solution. Let P_1 be the property "divisible by 2", let P_2 be the property "divisible by 3", and let P_3 be the property "divisible by 5". Since $1000 = 2 \cdot 500$, $1000 = 3 \cdot 333 + 1$, and $1000 = 5 \cdot 200$, then there are 500 integers between 1 and 1000 that are divisble by 2, 333 divisible by 3, and 200 divisible by 5. Since $1000 = 6 \cdot 166 + 4$, $1000 = 10 \cdot 100$, and $1000 = 15 \cdot 66 + 10$, there are 166 integers divisible by both 2 and 3 (so divisble by 6), 100 divisible by both 2 and 5 (so divisible by 10), and 66 divisible by both 3 and 5 (so divisible by 15). Since $1000 = 30 \cdot 33 + 10$, there are 33 integers divisible by 2,3, and 5 (so divisble by 30). By the Principle of Inclucion/Exclusion, there are

$$1000 - 500 - 333 - 200 + 166 + 100 + 66 - 33 = 266$$

integers from 1 to 1000 that are not divisible by 2,3, or 5. □

Exercise 1.2.2. A local donut shop has 4 types of donuts. Currently, there are 3 chocolate donuts, 4 vanilla donuts, 13 strawberry jelly donuts, and 2 crullers. How many different ways can you choose a dozen donuts?

Solution. Let x_1 be the number of chocolate donuts you choose, x_2 is the number of vanilla donuts you choose, x_3 is the number of strawberry donuts you choose, and x_4 is the number of crullers you choose. Then the question boils down to the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 12$$

with the condition $0 \leq x_1 \leq 3$, $0 \leq x_2 \leq 4$, $0 \leq x_3 \leq 13$ and $0 \leq x_4 \leq 2$.

Let the proerties be $P_1 := x_1 \geq 4$, $P_2 := x_2 \geq 5$, $P_3 := x_3 \geq 14$, and $P_4 := x_4 \geq 3$. Then we want to find the number of solutions where $x_1, x_2, x_3, x_4 \geq 0$ and none of P_1, P_2, P_3, nor P_4 are satisfied.

There are a total of $\binom{12+4-1}{4-1} = \binom{15}{3}$ total integer solutions.

Giving 4 1's to x_1, there are $\binom{8+4-1}{4-1} = \binom{11}{3}$ solution satisfying P_1. Giving 5 1's to x_2, there are $\binom{7+4-1}{4-1} = \binom{10}{3}$ solution satisfying P_2. Giving 14 1's to x_3, there are 0 solution satisfying P_3. Giving 3 1's to x_4, there are $\binom{9+4-1}{4-1} = \binom{12}{3}$ solution satisfying P_4.

Giving 4 1's to x_1 and 5 1's to x_2, there are $\binom{3+4-1}{4-1} = \binom{6}{3}$ solutions satisfying P_1 and P_2. Giving 4 1's to x_1 and 14 1's to x_3, there are 0 solutions satisfying P_1 and P_3. Giving 4 1's to x_1 and 3 1's to x_4, there are $\binom{5+4-1}{4-1} = \binom{8}{3}$ solutions satisfying P_1 and P_4. Giving 5 1's to x_2 and 14 1's to x_3, there are 0 solutions satisfying P_2 and P_3. Giving 5 1's to x_2 and 3 1's to x_4, there are $\binom{4+4-1}{3-1} = \binom{7}{3}$ solutions satisfying P_2 and P_4. Giving 14 1's to x_3 and 3 1's to x_4, there are 0 solutions satisfying P_3 and P_4.

Giving 4 1's to x_1, 5 1's x_2, and 3 1's to x_4, there are $\binom{0+4-1}{4-1} = \binom{3}{3} = 1$ There are no other solutions satisfying three of P_1, P_2, P_3, P_4, and no solutions satisfying all of P_1, P_2, P_3, P_4.

Therefore, by the Principle of Inclusion/Exclusion, the number of solutions satisfying none of P_1, P_2, P_3, P_4 is given by

$$\begin{aligned} &\binom{15}{3} - \binom{11}{3} - \binom{10}{3} - 0 - \binom{12}{3} + \binom{6}{3} + 0 + \binom{8}{3} + 0 + \binom{7}{3} + 0 - \binom{3}{3} - 0 - 0 - 0 - 0 - 0 + 0 \\ &= \binom{15}{3} - \binom{11}{3} - \binom{10}{3} - \binom{12}{3} + \binom{6}{3} + \binom{8}{3} + \binom{7}{3} - \binom{3}{3}. \end{aligned}$$

□

Exercise 1.2.3. In how many ways can the letters in UPPSALA be rearranged so that there are no occurrences of LAP, UP, SAP, or PAL?

Solution. Let P_1 be the property that LAP appears in a rearrangement, P_2 is the property that UP appears, P_3 is the property that SAP appears, and P_4 is the property that PAL appears.

- $N(\emptyset) = \binom{7}{2,2,1,1,1}$ since there are $\binom{7}{2,2,1,1,1}$ rearrangements of UPPSALA
- $N(\{1\}) = \binom{5}{1,1,1,1,1}$ since if we consider LAP as a single letter, there are $\binom{5}{1,1,1,1,1}$ rearrangements of $\underbrace{\text{LAP}}\text{UPSA}$.
- $N(\{2\}) = \binom{6}{2,1,1,1,1}$ since if we consider UP as a single letter, there are $\binom{6}{2,1,1,1,1}$ rearrangements of $\underbrace{\text{UP}}\text{PSALA}$.
- $N(\{3\}) = \binom{5}{1,1,1,1,1}$ since if we consider SAP as a single letter, there are $\binom{5}{1,1,1,1,1}$ rearrangements of $\underbrace{\text{SAP}}\text{UPAL}$.
- $N(\{4\}) = \binom{5}{1,1,1,1,1}$ since if we consider PAL as a single letter, there are $\binom{5}{1,1,1,1,1}$ rearrangements of $\underbrace{\text{PAL}}\text{UPSA}$.
- $N(\{1,2\}) = \binom{4}{1,1,1,1}$ since if we consider LAP and UP as single letters, there are $\binom{4}{1,1,1,1}$ rearrangements of $\underbrace{\text{LAP}}\underbrace{\text{UP}}\text{SA}$.
- $N(\{1,3\}) = \binom{3}{1,1,1}$ since if we consider LAP and SAP as single letters, there are $\binom{3}{1,1,1}$ rearrangements of $\underbrace{\text{LAP}}\underbrace{\text{SAP}}\text{U}$.
- $N(\{1,4\}) = \binom{3}{1,1,1}$. There are no rearrangements with LAP and PAL as seperate letters since there aren't enough L's, but if we consider $PALAP$ as a single letter, there are $\binom{3}{1,1,1}$ rearrangements of $\underbrace{\text{PALAP}}\text{US}$.
- $N(\{2,3\}) = \binom{4}{1,1,1,1}$ since if we consider UP and SAP as single letters, there are $\binom{4}{1,1,1,1}$ rearrangements of $\underbrace{\text{UP}}\underbrace{\text{SAP}}\text{LA}$.
- $N(\{2,4\} = 2\binom{4}{1,1,1,1}$ since there are $\binom{4}{1,1,1,1}$ rearrangements of $\underbrace{\text{UP}}\underbrace{\text{PAL}}\text{SA}$ and $\binom{4}{1,1,1,1}$ rearrangements of $\underbrace{\text{UPAL}}\text{PSA}$.
- $N(\{3,4\}) = \binom{4}{1,1,1,1} + \binom{3}{1,1,1}$ since there are $\binom{4}{1,1,1,1}$ rearrangements of $\underbrace{\text{SAP}}\underbrace{\text{PAL}}\text{UA}$ and $\binom{3}{1,1,1}$ rearrangements of $\underbrace{\text{SAPAL}}\text{UP}$
- $N(\{1,2,4\}) = 2$ since there are 2 rearrrangments of $\underbrace{\text{UPALAP}}\text{S}$.
- $N(\{2,3,4\}) = 2$ since there are 2 rearrangements of $\underbrace{\text{SAPAL}}\underbrace{\text{UP}}$.
- no other combinatrions of LAP, UP, SAP, and PAL can appear.

By the principle of Inclusion/Exclusion, the total number of rearrangements of UPPSALA that do not have occurences of LAP, UP, SAL, or PAL is given by

$$\begin{aligned}
\binom{7}{2,2,1,1,1} &- \binom{5}{1,1,1,1,1} - \binom{6}{2,1,1,1,1} - \binom{5}{1,1,1,1,1} - \binom{5}{1,1,1,1,1} \\
&+ \binom{4}{1,1,1,1} + \binom{3}{1,1,1} + \binom{3}{1,1,1} + \binom{4}{1,1,1,1} + 2\binom{4}{1,1,1,1} + \binom{4}{1,1,1,1} + \binom{3}{1,1,1} \\
&- 2 - 2 \\
&= 676.
\end{aligned}$$

$\square$

Exercise 1.2.4. A retiring Mathematics professor has 7 textbooks she wants to hand out to her last 4 graduate students. In how many ways can she distribute her textbooks so that every student gets at least 1?

Solution. This is simply the number of surjections from a set of 7 textbooks to a set of 4 graduate students (It is a surjection since every student gets at least 1 book). There are then

$$S(7,4) = \sum_{k=0}^{4} \binom{4}{k} (4-k)^7 = \binom{4}{0} 4^7 - \binom{4}{1} 3^7 + \binom{4}{2} 2^7 - \binom{4}{3} 1^7 + 0 = 8400.$$

$\square$

Exercise 1.2.5. There are 8 students sitting in a combinatorics class that only has 8 available seating places. During break, they leave to get coffee, and return to sit in different places. In how many ways can the students sit down after the break so that no student is seated where they were before the break.

Solution. This is the number of derangemets of 1,2,3,4,5,6,7,8, which is given by

$$\begin{aligned} d_8 &= \sum_{k=0}^{8} (-1)^k \binom{8}{k} (8-k)! \\ &= \binom{8}{0} 8! - \binom{8}{1} 7! + \binom{8}{2} 6! - \binom{8}{3} 5! + \binom{8}{4} 4! - \binom{8}{5} 3! + \binom{8}{6} 2! - \binom{8}{7} 1! + \binom{8}{8} 0! \\ &= 14833. \end{aligned}$$

$\square$

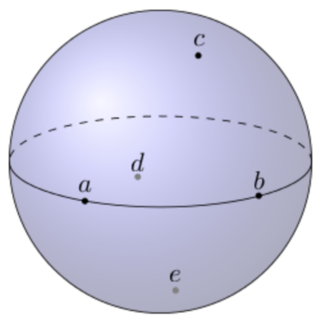
1.3 Pigeonhole Principle

Exercise 1.3.1. All of your socks are either black, white, or grey. How many socks do you need to pull from the dryer to guarantee that you have at least one pair of socks with matching colours?

Solution. If we let the socks be the pigeons, and the colours be the pigeonholes, then we have three pigeonholes and need 4 pigeons to guarantee that two pigeons are in one pigeonhole. Therefore, you need to pull 4 socks to guarantee that 2 have the same colour. $\square$

Exercise 1.3.2. Prove that no matter how 5 points are placed on a sphere, there is a hemisphere that contains at least 4 of the points.

Solution. Since there are 5 points, we can find two that are not antipodal (that is, you can find two points that do not form poles of the sphere). Draw a great circle through the two points (a great circle is a circle that cuts the sphere into two equal parts). There are now three remaining points left on the sphere which has now been seperated into two hemispheres. By the pigeonhole principle, two are on one of the hemisphere. Now slightly push the great circle to be off the two points, and there is now one hemisphere containing 4 points.



$\square$

For example above, we draw a great circle (in black) through a and b. By the pigeonhole principle, we can guarantee that one of the hemispheres has two points, for example the hemisphere containing d and e. Now we can redraw a new great circle (in red), and the hemisphere south of the red circle contains 4 points.

Exercise 1.3.3. Prove that no matter 19 integers are selected from the set $[35]$, two of the integers selected will sum to 36.

Solution. Consider the following 18 sets as pigeonholes:

$$\{1, 35\}, \{2, 34\}, \{3, 33\}, \{4, 32\}, \{5, 31\}, \ldots, \{17, 19\}, \{18\}.$$

The 19 integers chosen from $\{1, 2, \ldots, 35\}$ act as pigeons, and by the pigeonhole principle, some pigeonhole will contain two pigeons. Since $\{18\}$ can contain only one pigeon, the pigeonhole with two pigeons is one of the other sets, and the two integers in this set um to 36. $\square$

Exercise 1.3.4. Prove that no matter how 151 integers are chosen from the set $[300]$, there are two integers m and n so that $m|n$.

Solution. Consider the following 150 pigeonholes:

$$\{1, 1 \cdot 2^1, 1 \cdot 2^2, 1 \cdot 2^3, \dots, 1 \cdot 2^i, \dots, \}$$
$$\{3, 3 \cdot 2^1, 3 \cdot 2^2, 3 \cdot 2^3, \dots, 3 \cdot 2^i, \dots, \}$$
$$\{5, 5 \cdot 2^1, 5 \cdot 2^2, 5 \cdot 2^3, \dots, 5 \cdot 2^i, \dots, \}$$
$$\vdots$$
$$\{299, 299 \cdot 2^1, 299 \cdot 2^2, 299 \cdot 2^3, \dots, 299 \cdot 2^i, \dots, \}.$$

Every integer from [300] can be written as $t \cdot 2^m$ where t is an odd number, so every of the 151 integers chosen are in one of the pigeonholes above. By the pigeonhole principle, two of the numbers, say $x = t \cdot 2^n$ and $y = t \cdot 2^m$ with $n > m$ are in the same pigeonhole. Since

$$\frac{x}{y} = \frac{t \cdot 2^n}{t \cdot 2^m} = 2^{n-m}$$

is a whole number, then y divides x, that is $y|x$. □

Exercise 1.3.5. Elin is an engineering student who drinks a lot of coffee at Café Ångström. During the month of November, she drank at least one cup of coffee a day, but drank at most 45 coffees altogether. Prove that there is a span of consecutive days during which she drank exactly 14 coffees.

Solution. November has 30 days. Let x_i be the number of coffees Elin drank on November i'th, and let $y_i = x_1 + \cdots + x_i$ (so y_i is the number of coffees Elin drank in the first i days of November). Since each $x_i \geq 1$ (she drank at least one coffee a day), then

$$1 \leq y_1 < y_2 < \cdots < y_{30} \leq 45,$$

since she drank no more than 45 coffees. Also, by adding 14 to all the y_i's we have

$$15 \leq y_1 + 14 < y_2 + 14 < \cdots < y_{30} + 14 \leq 59.$$

The pigeons are the 60 numbers $y_1, y_2, \dots, y_{30}, y_1 + 14, y_2 + 14, \dots, y_{30} + 14,$, and the piegonholes are the 59 possible values $\{1, 2, \dots, 59\}$ these numbers can take. By the pigeonhole, two numbers are the same. Since $y_i < y_j$ and $y_i + 14 < y_j + 14$ for all $i < j$, the two numbers that are the same are of the form $y_i + 14$ and y_j for $i < j$. Then

$$14 = y_j - y_i = (x_1 + \cdots + x_j) - (x_1 + \cdots + x_i) = x_{i+1} + \cdots + x_j,$$

So from November i to November j, Elin drank exactly 14 coffees. □

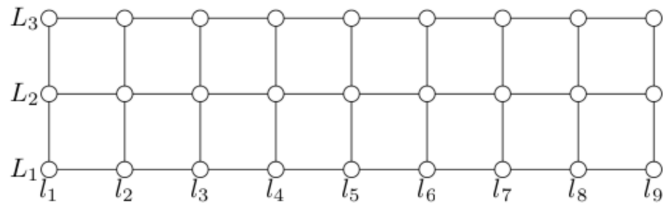
Exercise 1.3.6. Your neighbour is having a yard sale, with everything priced between 1kr and 100kr. Show that for no matter which 10 selected objects, two nonempty piles of objects can be made from the selected objects such that the price of the items in each pile sum to the same number.

Solution. Let S be any set of 10 items. There are a total of 2^10 possible different subsets A of S. We avoid two subsets,$\emptyset$ and S (since we are interesdted in non-empty piles, and if one pile contains all of S then the other contains nothing). So consider the remaining $2^10 - 2 = 1024 - 2 = 1022$ possible subsets; these will be our pigeons. As for the pigeonholes, the sum of the itms in any subset can range from 1 kr to $9 \cdot 100 = 900$ kr (no subset has 10 items). By the pigeonhole principle, two subsets $A, B \subset S$ have the same sum for the cost of the items. The only issue is that it is possible that $A \cap B \neq \emptyset$, in which case we can't make two seperate piles,

So let $C = A \cap B$ and look at $A' = A \setminus C$ and $B' = B \setminus C$, so remove the items that are in common in both A and B. Since we removed the same objects from both sets, A' and B' still contain items that sum to the same value, and $A' \cap B' = \emptyset$. These are our nonempty piles. □

Exercise 1.3.7. Consider a board of 8 square by 2 squares (so there are 16 squares in total). Suppose we draw coloured circles at each of the 27 corners of squares, each circle is either red or blue. Prove that there is some rectangle on the board with all 4 corners having the same colour.

Solution. Consider the grid as 2 rows and 8 columns of squares. This produces 3 horizontal lines and 9 vertical lines, making our 27 points of intersection that are coloured red or blue.



Each of the vertical lines has three circles which can be coloured red or blue. There are then $2^3 = 8$ different ways to colour the circles on each vertical line. Since there are 9 lines, by the pigeonhold principle, 2 of the vertical lines l_i, l_j have the same sequence of colours. There there are 3 circles and 2 colours, by the pigeonhole principle again, 2 of the circles in the sequence have the same colour, say on the horizontal lines L_m, L_n. Then the circles at $(l_i, L_m), (l_j, L_n), (l_i, L_m)$ and (l_j, L_n) all have the same colours, and make up the corners of a rectangle. $\square$

Generating Functions and Recurrence Relations

1.1 Generating Functions

Textbook readings

- From Keller + Trotter: Sections 8.1, 8.2, 8.5

Notation, Definitions, and Theorems

Here are some useful generating functions:

- $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$.
- $\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n$.
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$.
- $\frac{1}{(1-x)^n} = 1 + \binom{n}{n-1}x + \binom{n+1}{n-1}x^2 + \binom{n+2}{n-1}x^3 + \cdots = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k$.
- For $F(x) = \sum_{k=0}^{\infty} a_k x^k$ and $G(x) = \sum_{k=0}^{\infty} b_k x^k$, then $H(x) = F(x)G(x)$ is given by $H(x) = \sum_{k=0}^{\infty} c_k x^k$ where
$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots a_{k-1} b_1 + a_k b_0 = \sum_{i=0}^{k} a_i b_{k-i}.$$

Here are some useful exponential generating functions:

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$
- $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$

Exercises

Suggested exercises from textbooks

- From Keller + Trotter: Section 8.8, exercises 1 – 5, 7, 9, 14, 20, 21, 23.

Exercise 1.1.1. Find a closed form of the generating function for the sequence $\{a_k | k \geq 0\}$ given by

(a) $a_k = 3^k$

(b) $a_k = \frac{1}{3^k}$

(c) $a_k = \begin{cases} 0 & k = 0, 1, 2, 3, 4 \\ k - 4 & k \geq 5 \end{cases}$

(d) $a_k = \begin{cases} 2^k & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

Exercise 1.1.2. what is the coefficient of x^{12} in

(a) $(x^2 + x^4 + x^6)(x^6 + x^8 + x^{10})$

(b) $\frac{x^7}{1-(x/2)^2}$

(c) $\frac{1-x^{22}}{1-x}$

(d) $\frac{x^2}{(1-2x^2)^4}$

Exercise 1.1.3. What is the number of integer solutions to $x_1 + x_2 + x_3 = k$ with the restrictions $0 \leq x_1 \leq 3$, x_2 must be a multiple of 4, and $x_3 \geq 1$?

Exercise 1.1.4. Use generating functions to find the number of ways can 24 apples be distributed amongst 4 students so that every student gets at least 3 apples, but no more than 8?

Exercise 1.1.5. How many strings of length n are there consisting of $\{a, b, c, d\}$ so that the number of b's is even, the number of c's is odd, and d appears at least once.

1.2 Recurrence Relations

Textbook readings

- From Keller + Trotter: Sections 9.1 – 9.5

Notation, Definitions, and Theorems

Advancement Operators Suppose for the sequence $\{f_k | k \geq 0\}$ we have a recurrence of the form

$$c_0 f_{k+m} + c_1 f_{k+m-1} + c_2 f_{k+m-2} + \cdots + c_m f_k = 0.$$

Applying the advancement operator gives

$$p(A) f_k = (c_0 A^m + c_1 A^{m-1} + c_2 A^{m-2} + \cdots + c_k) f_k = 0.$$

Suppose $p(A) = (A - r_1)^{d_1} (A - r_2)^{d_2} \cdots (A - r_m)^{d_m}$. Then

$$\begin{aligned} f_k =& (a_{1,1} + k a_{1,2} + k^2 a_{1,3} + \cdots + k^{d_1 - 1} a_{1,d_1}) r_1^k + (a_{2,1} + k a_{2,2} + k^2 a_{2,3} + \cdots + k^{d_2 - 1} a_{2,d_1}) r_2^k + \\ & \cdots + (a_{m,1} + k a_{m,2} + k^2 a_{m,3} + \cdots + k^{d_m - 1} a_{1,d_m}) r_m^k. \end{aligned}$$

Exercises

Suggested exercises from textbooks

- From Keller + Trotter: Section 9.9, exercises 1–4, 6–9, 13.

Exercise 1.2.1. Solve the recurrence equation $f_0 = 2$, $f_1 = 5$, $f_k = 6 f_{k-1} - 8 f_{k-2}$, $k \geq 2$.

Exercise 1.2.2. Solve the recurrence equation $g_0 = -1$, $g_k = 3 g_{k-1} + (-1)^k + 1$, $k \geq 2$.

Exercise 1.2.3. Solve the recurrence equation $h_0 = 5$, $h_1 = 6$, $h_k = 4 h_{k-1} - 4 h_{k-2} + k - 1$, $k \geq 2$.

Exercise 1.2.4. How many binary strings of length n have no occurrences of 110?

Exercise 1.2.5. Recall the k'th Fibonacci number f_k defined recursively by $f_0 = 0, f_1 = 1$, and $f_k = f_{k-1} + f_{k-2}$ for $k \geq 2$. Find a closed form for f_k.

1.3 Further Examples

Textbook readings

- From Keller + Trotter: Sections 8.3 – 8.5, 9.6.

Notation, Definitions, and Theorems

- For any real number x and any positive integer k, the falling fatorial of x is

$$(x)_k = x(x-1)\cdots(x-k+1),$$

  while $(x)_0 = 1$. The textbook uses the notation $P(x,k)$.

- For any real number x and any integer $k \geq 0$, the generalized binomial coefficient is defined as

$$\binom{x}{k} = \frac{(x)_k}{k!}.$$

Theorem 1.3.1 (Newton's Binomial Theorem, or The Binomial Series)**.** *For all real numbers $p \neq 0$, then*

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n.$$

Exercises

Suggested exercises from textbooks

- From Keller + Trotter: Section 8.8, exercises 16–19.

Exercise 1.3.1. Solve exercise 1.2.1 using generating functions.

Exercise 1.3.2. Solve exercise 1.2.2 using generating functions.

Exercise 1.3.3. Solve exercise 1.2.3 using generating functions.

Tuesday, February 16th

Generating Functions

- Denote a sequence $a_0, a_1, a_2, \ldots$ by $\{a_k\}_{k=0}^{\infty}$ (or $\{a_k \mid k \geq 0\}$)
- For a sequence $\{a_k\}_{k=0}^{\infty}$, we associate a function

$$F(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

- $F(x)$ is called the generating function for $\{a_k\}_{k=0}^{\infty}$.
- Notice $F(0) = a_0$, but usually we don't care about evaluting F at specific values of x.
- Usually not concerned about the convergence of $\sum_{k=0}^{\infty} a_k x^k$, but for this class all generating functions we look at will have positive radii of convergence.

Example: For a fixed integer (for now) $n \geq 0$, consider the sequence $\{a_k\}_{k=0}^{\infty}$ given by $a_k = \binom{n}{k}$ for $k = 0, 1, \ldots, n$, $a_k = 0$ for $k > n$.

From the Binomial Theorem, the generating function for $\{a_k\}_{k=0}^{\infty}$ is

$$F(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

Example: For a fixed integer $n \geq 0$, consider the sequence $\{a_k\}_{k=0}^{\infty}$ given by $a_k = 1$ for $k = 0, 1, \ldots, n$, and $a_k = 0$ for $k > n$.

Then the generating function is given by $F(x) = \sum_{k=0}^{\infty} a_k x^k = 1 + x + x^2 + \cdots + x^n$

But we want $F(x)$ in a form that's easier to work with. Notice that $1 - x^{n+1} = (1 + x + x^2 + \cdots + x^n) - (x + x^2 + \cdots + x^{n+1}) = (1-x)F(x)$, so

$$F(x) = \frac{1 - x^{n+1}}{1 - x}.$$

Example: Let $\{a_K\}_{K=0}^{\infty}$ be given by $a_K=1$ for all $K\geq 0$. So $F(x)=\sum_{K=0}^{\infty} x^K$. We have that $1=(1+x+x^2+\cdots)-(x+x^2+x^3+\cdots)=(1-x)F(x)$, so after rearranging, $F(x)=\frac{1}{1-x}$. You may have seen this as the McLaurin (or Taylor) series of $\frac{1}{1-x}$.

Example: Let's give an analytic argument to find the generating function of the sequence $\{a_K\}_{K=0}^{\infty}$ given by $a_K=\binom{n+K-1}{n-1}=\binom{n+K-1}{K}$.

Claim: $\sum_{K=0}^{\infty}\binom{n+K-1}{n-1}x^K=\frac{1}{(1-x)^n}$.

Base Cases: We saw $\frac{1}{1-x}=\sum_{K=0}^{\infty}x^K=\sum_{K=0}^{\infty}\binom{K}{0}x^K$, so the claim holds for $n=1$.

Let $F(x)=\frac{1}{1-x}$, then by the addition rule for derivatives,

$$\frac{1}{(1-x)^2}=F'(x)=\sum_{K=0}^{\infty}(x^K)'=\sum_{K=1}^{\infty}Kx^{K-1}=\sum_{K=0}^{\infty}(K+1)x^K=\sum_{K=0}^{\infty}\binom{K+1}{1}x^K=\sum_{K=0}^{\infty}\binom{2+K-1}{2-1}x^K,$$

So the claim holds for $n=2$.

Inductive Step: Let $m\geq 1$, and assume $G(x):=\sum_{K=0}^{\infty}\binom{m+K-1}{m-1}x^K=\frac{1}{(1-x)^m}$

Then $G'(x)=\frac{m}{(1-x)^{m+1}}$, so

$$\frac{1}{(1-x)^{m+1}}=\frac{1}{m}G'(x)=\frac{1}{m}\sum_{K=0}^{\infty}\left(\binom{m+K-1}{m-1}x^K\right)'=\sum_{K=1}^{\infty}\frac{K}{m}\binom{m+K-1}{m-1}x^{K-1}=\sum_{K=0}^{\infty}\frac{(K+1)}{m}\binom{m+K}{m-1}x^K$$

Since $\frac{K+1}{m}\binom{m+K}{m-1}=\frac{\cancel{K+1}}{m}\frac{(m+K)!}{(m-1)!\,(\cancel{K+1})!}=\frac{(m+K)!}{m!\,K!}=\binom{m+K}{m}$, we get that

$$\frac{1}{(1-x)^{m+1}}=\sum_{K=0}^{\infty}\binom{m+K}{m}x^K,$$ concluding the inductive step.

By PMI, $\frac{1}{(1-x)^n}=\sum_{K=0}^{\infty}\binom{n+K-1}{n-1}x^K$ for all $n\geq 1$.

Why use Generating Functions?

- It allows us to use analysis to solve combinatorial problems.
- In this course, we will only look at basic uses, like function multiplication.

Look at the generating functions $F(x)=\sum_{k=0}^{\infty} a_k x^k$ and $G(x)=\sum_{k=0}^{\infty} b_k x^k$. What is the k'th term of $H(x)=F(x)G(x)=\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} b_k x^k\right)$?

After multiplying, gather all of the x^k terms. How many ways can we get x^k? Look at all $a_j x^j b_{k-j} x^{k-j} = a_j b_{k-j} x^k$. So the k'th term of $H(x)$ is $\sum_{j=0}^{k} a_j b_{k-j} x^k$, which the sum of all the ways of choosing terms from $F(x)$ and $G(x)$ whose product is a multiple of x^k.

In general, let $F_1(x)=\sum_{k=0}^{\infty} a_{1,k} x^k$, $F_2(x)=\sum_{k=0}^{\infty} a_{2,k} x^k$, ..., $F_d(x)=\sum_{k=0}^{\infty} a_{d,k} x^k$.
The k'th term of $F_1(x)F_2(x)\cdots F_d(x)$ is the sum of all the ways of choosing terms from $F_1(x),\dots,F_d(x)$ whose product is a multiple of x^k, so it's given by

$$a_k x^k = \sum_{k_1+\cdots+k_d=k} a_{1,k_1} x^{k_1} a_{2,k_2} x^{k_2} \cdots a_{d,k_d} x^{k_d} = \sum_{k_1+\cdots+k_d=k} a_{1,k_1} a_{2,k_2} \cdots a_{d,k_d} x^k.$$

Example: Using generating functions, find the number of integer solutions to

$$x_1+x_2+x_3+x_4+x_5=k \qquad x_1,x_2,x_3,x_4,x_5 \geq 0.$$

Let $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$, $\{c_k\}_{k=0}^{\infty}$, $\{d_k\}_{k=0}^{\infty}$, $\{e_k\}_{k=0}^{\infty}$ be the number of solutions to $x_1=k, x_1\geq 0$, $x_2=k, x_2\geq 0$, $x_3=k, x_3\geq 0$, $x_4=k, x_4\geq 0$, $x_5=k, x_5\geq 0$ respectively. Clearly $a_k=b_k=c_k=d_k=e_k=1$ for all $k\geq 0$. Now let

$$A(x)=B(x)=C(x)=D(x)=E(x)=\frac{1}{1-x}$$

be the generating functions for the sequences above, respectively.

Let $S(x) = A(x)B(x)C(x)D(x)E(x)$. Then the k'th coefficient of $S(x)$ is the sum over all solutions to $x_1 = k_1$, $x_2 = k_2$, $x_3 = k_3$, $x_4 = k_4$, $x_5 = k_5$ such that $k_1+k_2+k_3+k_4+k_5 = k$, which is the number of solutions to $x_1+x_2+x_3+x_4+x_5 = k$, $x_1, x_2, x_3, x_4, x_5 \geq 0$. Since $S(x) = \frac{1}{1-x} \cdot \frac{1}{1-x} \cdot \cdots \cdot \frac{1}{1-x} = \frac{1}{(1-x)^5}$, we know that the k'th coefficient of $S(x)$ is given by

$$\binom{5+k-1}{5-1} = \binom{k+4}{4}.$$

<u>Example</u>: What is the number of integer solutions to

$$x_1 + x_2 + x_3 = k \text{ such that } 0 \leq x_1 \leq 5, \ x_2 \text{ is even}, \ x_3 \text{ is a multiple of } 6?$$

Let $\{a_k\}_{k=0}^{\infty}$ such that a_k is the number of solutions to $x_1 = k$, $0 \leq x_1 \leq 5$. Then

$$F(x) = \sum_{k=0}^{\infty} a_k x^k = 1 + x + x^2 + x^3 + x^4 + x^5 = \frac{1-x^6}{1-x}$$

Let $\{b_k\}_{k=0}^{\infty}$ such that b_k is the number of solutions to $x_2 = k$, x_2 is even. Then $b_k = 1$ when k is even and $b_k = 0$ otherwise and

$$G(x) = \sum_{k=0}^{\infty} b_k x^k = 1 + x^2 + x^4 + x^6 + \cdots = \sum_{k=0}^{\infty} (x^2)^k = \frac{1}{1-x^2}$$

$$\sum_{k=0}^{\infty} y^k = \frac{1}{1-y}$$

replace y with x^2

Let $\{c_k\}_{k=0}^{\infty}$ such that c_k is the number of solutions to $x_3 = k$, x_3 multiple of 6. Then $c_k = 1$ when k is a multiple of 6, $c_k = 0$ otherwise, and

$$H(x) = \sum_{k=0}^{\infty} c_k x^k = 1 + x^6 + x^{12} + x^{18} + \cdots = \sum_{k=0}^{\infty} (x^6)^k = \frac{1}{1-x^6}.$$

Let s_k be the number of solutions to $x_1+x_2+x_3 = k$ with the conditions for x_1, x_2, x_3 above. Then

$$S(x) = \sum_{k=0}^{\infty} s_k x^k = F(x)G(x)H(x) = \frac{\cancel{1-x^6}}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{\cancel{1-x^6}}$$

$$= \frac{1}{(1-x)(1-x^2)} = \frac{1}{(1-x)^2(1+x)}$$

Since $1-x^2 = (1-x)(1+x)$

Recall: Partial fraction Decomposition

There's plenty of videos and tutorials online for this

Can you split $\frac{1+x^2}{(x^2-3)(x-2)^2}$ into a sum of fractions?

Suppose
$$\frac{1+x^2}{(x^2-3)(x-2)^2} = \frac{A+Bx}{x^2-3} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

Give the numerator a polynomial of degree one less than the denominator

Since $x-2$ appears twice, we add $\frac{C}{x-2}$ and $\frac{D}{(x-2)^2}$.

If you have something appearing m times, then add m terms with increasing powers on the denominator.

Multiplying things out, we see that

$$\frac{1+x^2}{(x^2-3)(x-2)^2} = \frac{(A+Bx)(x-2)^2}{(x^2-3)(x-2)^2} + \frac{C(x-2)(x^2-3)}{(x^2-3)(x-2)^2} + \frac{D(x^2-3)}{(x^2-3)(x-2)^2},$$

So $1+x^2 = Ax^2 - 4Ax + 4A + Bx^3 - 4Bx^2 + 4Bx + Cx^3 - 2Cx^2 - 3Cx + 6C + Dx^2 - 3D$

Solve the system of equations

$$\left.\begin{aligned} Bx^3 + Cx^3 &= 0x^3 \\ Ax^2 - 4Bx^2 - 2Cx^2 + Dx^2 &= 1x^2 \\ -4Ax + 4Bx - 3Cx &= 0x \\ 4A + 6C - 3D &= 1 \end{aligned}\right\} \quad \begin{aligned} A &= 28, \\ B &= 16 \\ C &= -16 \\ D &= 5 \end{aligned}$$

So $\frac{1+x^2}{(x^2-3)(x-2)^2} = \frac{28+16x}{x^2-3} - \frac{16}{x-2} + \frac{5}{(x-2)^2}$

For our example above, $S(x) = \frac{1}{(1-x)^2(1+x)} = \frac{A}{1+x}\cdot\frac{(1-x)^2}{(1-x)^2} + \frac{B}{1-x}\cdot\frac{(1-x)(1+x)}{(1-x)(1+x)} + \frac{C}{(1-x)^2}\cdot\frac{(1+x)}{(1+x)}$, So

$1 = A(1-x)^2 + B(1+x)(1-x) + C(1+x) = A - 2Ax + Ax^2 + B - Bx^2 + C + Cx$

Solving
$$\begin{aligned} Ax^2 - Bx^2 &= 0 \Rightarrow A = B \\ -2Ax + Cx &= 0 \Rightarrow C = 2A \\ A + B + C &= 1 \Rightarrow A + A + 2A = 1 \Rightarrow 4A = 1 \Rightarrow A = \tfrac{1}{4},\ B = \tfrac{1}{4},\ C = \tfrac{1}{2}. \end{aligned}$$

So $S(x) = \frac{1}{4}\left(\frac{1}{1-(-x)}\right) + \frac{1}{4}\left(\frac{1}{1-x}\right) + \frac{1}{2}\left(\frac{1}{(1-x)^2}\right)$

$= \frac{1}{4}\sum_{k=0}^{\infty}(-x)^k + \frac{1}{4}\sum_{k=0}^{\infty}x^k + \frac{1}{2}\sum_{k=0}^{\infty}(k+1)x^k$

So the number of solutions is given by

$s_k = \frac{(-1)^k}{4} + \frac{1}{4} + \frac{k+1}{2}$

Exponential Generating Functions:

The generating functions we saw so far are useful for "combinations-like" sequences, since these sequence don't grow too fast. But for "permutation-like" sequences which grow faster, we use a different type of generating function.

For a sequence $\{a_K\}_{K=0}^{\infty}$, define the **Exponential Generating Function** to be $F(x) = \sum_{K=0}^{\infty} \frac{a_K}{K!} x^K$.

Example: If $\{a_K\}_{K=0}^{\infty}$ such that $a_K = 1$ for all $K \geq 0$, then

$$F(x) = \sum_{K=0}^{\infty} \frac{a_K}{K!} x^K = \sum_{K=0}^{\infty} \frac{x^K}{K!} = e^x$$

Example: Recall $P(n,K) = \frac{n!}{(n-K)!}$. For a fixed n, let $a_K = P(n,K)$ for $0 \leq K \leq n$, and $a_K = 0$ for $K > n$. Then the exponential generating function for $\{a_K\}_{K=0}^{\infty}$ is

$$F(x) = \sum_{K=0}^{\infty} \frac{a_K}{K!} x^K = \sum_{K=0}^{n} \frac{n!}{K!(n-K)!} x^K = \sum_{K=0}^{n} \binom{n}{K} x^K = (1+x)^n$$

gen. funct. for $\binom{n}{K}$ AND exp. gen. funct. for $P(n,K)$

Example: Let $\{a_K\}_{K=0}^{\infty}$ given by $a_K = \begin{cases} 0 & K \text{ even} \\ 1 & K \text{ odd} \end{cases}$. Then

$$F(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)$$

$$= \frac{1}{2}e^x - \frac{1}{2}e^{-x} = \frac{e^x - e^{-x}}{2}.$$

Example: Let $\{a_K\}_{K=0}^{\infty}$ given by $a_K = \begin{cases} 1 & K \text{ even} \\ 0 & K \text{ odd} \end{cases}$. Then

$$F(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)$$

$$= \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{e^x - e^{-x}}{2}.$$

Example: Exercise 9.24 from Keller + Trotter textbook.

How many strings of length n from $\{a,b,c,d\}$ where there is at least 1 a and the number of c's is odd.

Let $a_k :=$ # strings of length k of a's with at least 1 a, so $a_0 = 0$, $a_k = 1$ for $k \geq 1$

$$A(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} - 1 = e^x - 1.$$

Let $b_k :=$ # strings of length k of b's, so $b_k = 1$

$$B(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

Let $c_k :=$ # strings of length k of odd c's, so $c_k = \begin{cases} 0 & k \text{ even} \\ 1 & k \text{ odd} \end{cases}$

$$C(x) = \sum_{k=0}^{\infty} \frac{c_k}{k!} x^k = \frac{e^x - e^{-x}}{2}$$

Let $d_k :=$ # strings of length k of d's, $D(x) = e^x$

Let $S(x)$ be the exp. gen. func. for # of strings that we want,

$$S(x) = A(x)B(x)C(x)D(x) = (e^x - 1)e^x\left(\frac{e^x - e^{-x}}{2}\right)e^x$$

$$= \frac{e^{4x} - e^{2x}}{2} - \frac{e^{3x} - e^x}{2}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{(4x)^k}{k!} - \frac{1}{2}\sum_{k=0}^{\infty} \frac{(2x)^k}{k!} - \frac{1}{2}\sum_{k=0}^{\infty} \frac{(3x)^k}{k!} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

So the number of strings of length k is

$$\frac{4^k}{2} - \frac{2^k}{2} - \frac{3^k}{2} + \frac{1}{2}$$

Tuesday, February 23rd

Recurrence Relations

Recursive Definition

We have explicitely defined a lot of types of numbers. For example, we defined:

$$n! := n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1$$

$$P(n,k) := \frac{n!}{(n-k)!}$$

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

$$C_n := \frac{1}{n+1}\binom{2n}{n}$$

Sometimes, an explicit formula is difficult (or impossible) to find, so it's better to give a recursive definition:

Example: $n!$ can be defined recursively by

$$\begin{cases} 1! := 1, \text{ and} \\ n! := n(n-1)!, \ n \geq 2 \end{cases}$$

Example: (Fibonacci Sequence) define $F_n, n \geq 1$ by

$$\begin{cases} F_1 = 1, \\ F_2 = 1, \text{ and} \\ F_n = F_{n-1} + F_{n-2}, \ n \geq 3 \end{cases}$$

This yields the famous Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

This sequence is related to the famous golden ratio φ by

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} =: \varphi$$

Example: Instead of using the explicit formula for $\binom{n}{k}$ to prove Pascal's identity, we can use Pascal's identity to define the binomial coefficients:

$$\begin{cases} \binom{0}{0} = 1 \\ \binom{n}{0} = 1 \\ \binom{n}{n} = 1 \\ \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}, \quad n \geq r \geq 1. \end{cases}$$

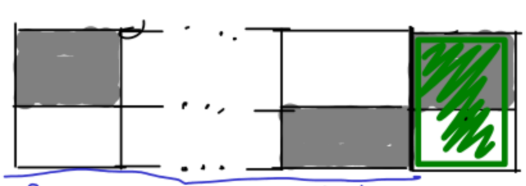
Solving Problems Recursively

To solve several combinatorial problems, we first start by finding a recursive formula for the solution.

Example: Consider a $n \times 2$ checkerboard, and a set of 1×2 and 2×1 dominoe pieces. In how many ways can you cover the checkerboard with dominoe pieces?

Let t_n be the number of ways of tiling a $2\times n$ checkerboard. Then $t_1 = 1$ and $t_2 = 2$:

For $n \geq 3$, we can tile a $2\times n$ checkerboard by either taking a $2\times(n-1)$ tiling and adding a vertical tile, or taking a $2\times(n-2)$ tiling and adding 2 horizontal tiles.



Some $2\times(n-1)$ tiling

Some $2\times(n-2)$ tiling

So we get that $t_n = t_{n-1} + t_{n-2}$. So t_n has the solution

$$\begin{cases} t_1 = 1 \\ t_2 = 2 \\ t_n = t_{n-1} + t_{n-2}, \quad n \geq 3 \end{cases}$$

Example: How many $\overset{\text{3-ary}}{\text{ternary}}$ strings of length n do not contain 12 as a substring?

Let s_n be the number of such strings. Then $s_0 = 1$ (the empty string), $s_1 = 3$ (0, 1, 2), and $s_2 = 3^2 - 1 = 8$ (00, 01, 02, 10, 11, 20, 21, 22).

For $n \geq 3$, we can create such a string by taking a 12-avoiding string s of length $n-1$ and adding 0, 1, or 2 at the beginning. But if s started with 2 and we add 1, then we get 12 at the beginning, so we need to remove these strings. We can make a 12-avoiding string of length $n-1$ that starts with 2 by taking a 12-avoiding string of length $n-2$ and adding a 2 at the beginning.

So $s_n = \underbrace{3s_{n-1}}_{\substack{\text{add 0,1,2} \\ \text{at the beginning} \\ \text{of string of length} \\ n-1}} - \underbrace{s_{n-2}}_{\substack{\text{strings of length} \\ n-1 \text{ starting} \\ \text{with 2.}}}$, so

$$\begin{cases} s_0 = 1 \\ s_1 = 3 \\ s_2 = 8 \\ s_n = 3s_{n-1} - s_{n-2}, \quad n \geq 3. \end{cases}$$

Solving Recurrence Relations

If we have a guess of a closed form, we might be able to solve a recurrence relation using induction.

Example: Look at the recurrence relation

$$\begin{cases} f_1 = 2 \\ f_n = 3f_{n-1} + 1. \end{cases}$$

The first few terms are 2, 7, 22, 67, 202, ...

You might be able to guess

You might be able to guess $f_n = \frac{1}{6}(5 \cdot 3^n - 3)$

Base Case: Let $n=1$, $f_1 = 2 = \frac{1}{6}(5\cdot 3^1 - 3) = \frac{1}{6}(12)$, So base case is true.

Inductive Step: Let $k \geq 1$, and assume $f_k = \frac{1}{6}(5\cdot 3^k - 3)$. Then

$$f_{k+1} \underset{\text{by definition}}{=} 3f_k + 1 \underset{\text{by I.H.}}{=} 3\left(\frac{1}{6}(5\cdot 3^k - 3)\right) + 1 = \frac{1}{6}(5\cdot 3^{k+1} - 9) + \frac{6}{6} = \frac{1}{6}(5\cdot 3^{k+1} - 3)$$

So the inductive step holds.

By PMI, $f_n = \frac{1}{6}(5\cdot 3^n - 3)$ for $n \geq 1$.

But making a guess might be difficult, and the induction proof may not be so simple.

Advancement Operator

Let $\{f_k\}_{k=0}^{\infty}$ be a sequence. We define the Advancement Operator A by $Af_k = f_{k+1}$. Applying the operator several times, we get $A^p f_k = \underbrace{A(A(A(\cdots(Af_k)\cdots)}_{p \text{ times}} = f_{k+p}$.

Example: Suppose $\{s_k\}_{k=0}^{\infty}$ is defined by $s_0 = 3$, $s_k = 2s_{k-1}$ for $k \geq 1$.
We can immediately see that $s_k = 3\cdot 2^k$. With the advancement operator, we see that $As_k = s_{k+1} = 2s_k$. Rearrange as

$$0 = s_{k+1} - 2s_k = As_k - 2s_k = (A-2)s_k$$

$(A-2)=0$ has root $A=2$, which tells us $s_k = a\cdot 2^k$ for some constant a.
Use $s_0 = a\cdot 2^0 = 3$ to get $a=3$, so $s_k = 3\cdot 2^k$

(like $y' = 2y \Leftrightarrow y = a\cdot e^{2t}$; $y(0) = 3 = ae^0 \Rightarrow y = 3\cdot 2^t$)

General Case (Homogeneous Case): Suppose we have a recurrence of the form

$$c_0 f_{k+m} + c_1 f_{k+m-1} + c_2 f_{k+m-2} + \cdots + c_m f_k = 0,$$

with $c_0, c_m \neq 0$. This is called a homogeneous recurrence

Since it is equal to 0. We then rewrite this relation as

$$p(A)f_K = (c_0A^m + c_1A^{m-1} + c_2A^{m-2} + \cdots + c_{m-1}A + c_m)f_K = 0,$$

so $p(A)$ is a polynomial. Suppose $p(A)$ has distinct roots $r_1, r_2, \ldots, r_m$, so $p(A) = c_0A^m + c_1A^{m-1} + \cdots + c_m = (A-r_1)(A-r_2)\cdots(A-r_m)$.

Then $f_K = a_1r_1^K + a_2r_2^K + \cdots + a_mr_m^K$. We then use

$$f_0, f_1, f_2, \ldots, f_{m-1}$$

to solve for $a_1, a_2, a_3, \ldots, a_m$.

Example: Look at the sequence $\{t_K\}_{K=0}^{\infty}$ defined by

$$\begin{cases} t_0 = 4 \\ t_1 = 5 \\ t_K = t_{K-1} + 2t_{K-2}, \ K \geq 2. \end{cases}$$

We rewrite the recursion $t_{K+2} = t_{K+1} + 2t_K \iff t_{K+2} - t_{K+1} - 2t_K = 0$, then apply the advancement operator

$$0 = t_{K+2} - t_{K+1} - 2t_K = A^2t_K - At_K - 2t_K = (A^2-A-2)t_K = (A-2)(A+1)t_K.$$

Therefore, $t_K = a_1\underbrace{2^K + a_2(-1)^K}_{\text{roots of } A^2-A-2}$. Since

$$t_0 = a_1 \cdot 2^0 + a_2 \cdot (-1)^0 = a_1 + a_2 = 4$$

$$t_1 = a_1 \cdot 2^1 + a_2(-1)^1 = 2a_1 - a_2 = 5$$

Solve this system of equations to get $a_1 = 3$, $a_2 = 1$, so

$$t_K = 3 \cdot 2^K + (-1)^K.$$

General Case (Homogeneous Case with repeated roots): Now suppose $p(A)$ has repeated roots, i.e.,

$$p(A) = (A-r_1)^{d_1}(A-r_2)^{d_2}\cdots(A-r_m)^{d_m}.$$

Then

$$f_K = (a_{1,1} + Ka_{1,2} + K^2a_{1,3} + \cdots + K^{d_1-1}a_{1,d_1})r_1^K + (a_{2,1} + Ka_{2,2} + \cdots + K^{d_2-1}a_{2,d_2})r_2^K + \cdots$$
$$+ (a_{m,1} + Ka_{m,2} + \cdots + K^{d_m-1}a_{m,d_m})r_m^K.$$

Example: Let $\{h_k\}_{k=0}^{\infty}$ be the sequence defined by

$$\begin{cases} h_0 = 1 \\ h_1 = 3 \\ h_2 = 29 \\ h_k = h_{k-1} + 8h_{k-2} - 12h_{k-3}, \quad k \geq 3. \end{cases}$$

$h_{k+3} = h_{k+2} + 8h_{k+1} - 12h_k, \quad k \geq 0$

Rewrite the recursion as

$$0 = h_{k+3} - h_{k+2} - 8h_{k+1} + 12h_k = A^3h_k - A^2h_k - 8Ah_k + 12h_k = (A^3 - A^2 - 8A + 12)h_k = (A-2)^2(A+3)h_k$$

Therefore, $h_k = a_1 \cdot 2^k + a_2 \cdot k \cdot 2^k + a_3(-3)^k$. Since

$$h_0 = a_1 \cdot 2^0 + a_2 \cdot 0 \cdot 2^0 + a_3(-3)^0 = a_1 + a_3 = 1$$
$$h_1 = a_1 \cdot 2^1 + a_2 \cdot 1 \cdot 2^1 + a_3(-3)^1 = 2a_1 + 2a_2 - 3a_3 = 3$$
$$h_2 = a_1 \cdot 2^2 + a_2 \cdot 2 \cdot 2^2 + a_3(-3)^2 = 4a_1 + 8a_2 + 9a_3 = 29$$

Solving the system of equations gives $a_1 = 1$, $a_2 = 2$, $a_3 = 1$, so

$$h_k = 2^k + 2k \cdot 2^k + (-3)^k = (1+2k)2^k + (-3)^k.$$

General Case (NonHomogeneous Case): Consider a recurrence of the form

$$c_0 f_{k+m} + c_1 f_{k+m-1} + c_2 f_{k+m-2} + \cdots + c_m f_k = g_k$$

for a sequence g_k. Suppose after applying the advancement operator, this becomes $p(A)f_k = g_k$. Let f_k' be a solution to $p(A)f_k' = 0$ and f_k'' is any solution to $p(A)f_k'' = g_k$. Then any solution to $p(A)f_k = g_k$ is of the form $f_k = f_k' + f_k''$.

called a particular solution

Example: Let $\{s_k\}_{k=0}^{\infty}$ be defined as

$$\begin{cases} s_0 = 0 \\ s_1 = 2 \\ s_k = 4s_{k-1} + 12s_{k-2} + 3k - 4 \end{cases}$$

Rewrite the recursion $s_{k+2} = 4s_{k+1} + 12s_k + \underbrace{3(k+2) - 4}_{=3k+2}$, so

$$3k + 2 = s_{k+2} - 4s_{k+1} - 12s_k = A^2s_k - 4As_k - 12 = (A^2 - 4A - 12)s_k = (A+2)(A-6)s_k, \text{ so}$$

$s_k' = a_1(-2)^k + a_2 6^k$ is a solution to $(A+2)(A-6)s_k' = 0$

To find a particular solution to $(A+2)(A-6)s''_k = 3k+2$, a good guess is $s''_k = bk+c$ for some b, c. Then

$$3k+2 = (A+2)(A-6)s''_k = (A^2-4A-12)(bk+c)$$
$$= A^2(bk+c) - 4A(bk+c) - 12(bk+c)$$
$$= (b(k+2)+c) - 4(b(k+1)+c) - 12(bk+c)$$
$$= -15bk - 15b - 15c \qquad b = \frac{-1}{5}$$

So $-15bk = 3k \Rightarrow b = \frac{-3}{15} = \frac{-1}{5}$, and $2 = -15b - 15c = -15c + 3 \Rightarrow c = \frac{1}{15}$.
So $s''_k = bk + c = \frac{-1}{5}k + \frac{1}{15}$ is a particular solution. So

$$s_k = s'_k + s''_k = a_1(-2)^k + a_2 6^k - \frac{1}{5}k + \frac{1}{15}.$$ Since

$$s_0 = a_1(-2)^0 + a_2 \cdot 6^0 - \frac{1}{5}(0) + \frac{1}{15} = 0 \Rightarrow a_1 + a_2 = \frac{-1}{15}$$
$$s_1 = a_1(-2)^1 + a_2 6^1 - \frac{1}{5}(1) + \frac{1}{15} = 2 \Rightarrow -2a_1 + 6a_2 = \frac{32}{15}$$

Solve

$2 - \frac{1}{15} + \frac{1}{5}$

Solving the system yields $a_1 = \frac{-19}{60}$, $a_2 = \frac{1}{4}$, so

$$s_k = \frac{-19}{20}(-2)^k + \frac{1}{4}\cdot 6^k - \frac{1}{5}\cdot k + \frac{1}{15}.$$

Particular Solution "Good Guesses"

- If g_k is a polynomial, f''_k be a polynomial of the same degree

  ex: $g_k = k^2+3k+2$, let $f''_k = ak^2+bk+c$

- If g_k is exponential, f''_k be exponential with the same base

  ex: $g_k = 3^k+1$, let $f''_k = a\cdot 3^k + b$

Thursday, February 25

Generating Functions + Recurrence Relations ctd.

Solving Recurrence Relations with Generating Functions.

- Define a generating function for the sequence
- Use the recurrence to find a closed form of the generating function
- Use the closed form of the gen. funct. to get a closed form of the sequence.

- Example: Use generating functions to find a closed form for $\{r_K\}_{K=0}^{\infty}$ defined by

$$\begin{cases} r_0 = 1 \\ r_1 = 3 \\ r_K = 6r_{K-2} - r_{K-1}, \quad K \geq 2. \end{cases}$$

Define $R(x) = \sum_{K=0}^{\infty} r_K x^K$. Then

$$R(x) = \sum_{K=0}^{\infty} r_K x^K = r_0 + r_1 x + \sum_{K=2}^{\infty} r_K x^K = 1 + 3x + \sum_{K=2}^{\infty} (6r_{K-2} - r_{K-1}) x^K = 1 + 3x + 6\sum_{K=2}^{\infty} r_{K-2} x^K - \underbrace{\sum_{K=2}^{\infty} r_{K-1} x^K}_{= \sum_{K=1}^{\infty} r_K x^{K+1} = x\sum_{K=1}^{\infty} r_K x^K = x\left(\sum_{K=0}^{\infty} r_K x^K - r_0\right)}$$

$$= 1 + 3x + 6x^2 \sum_{K=0}^{\infty} r_K x^K - x\left(\sum_{K=0}^{\infty} r_K x^K - \overset{1}{\overset{\|}{r_0}}\right) = 1 + 4x + (6x^2 - x)R(x)$$

So $R(x) = 1 + 4x + (6x^2 - x)R(x) \Rightarrow R(x)(1 + x - 6x^2) = 1 + 4x$

$$\Rightarrow R(x) = \frac{1+4x}{1+x-6x^2} = \frac{1+4x}{(1+3x)(1-2x)}$$

Then by solving the partial fraction decomposition,

$$\frac{1+4x}{(1+3x)(1-2x)} = \frac{A}{1+3x} + \frac{B}{1-2x} \Rightarrow \left.\begin{array}{l} A + B = 1 \\ -2Ax + 3Bx = 4x \end{array}\right\} \begin{array}{l} A = \frac{-1}{5} \\ B = \frac{6}{5}. \end{array}$$

we get $R(x) = \frac{-1}{5}\underbrace{\left(\frac{1}{1+3x}\right)}_{\frac{1}{1-(-3x)}} + \frac{6}{5}\left(\frac{1}{1-2x}\right) = \frac{-1}{5}\sum_{K=0}^{\infty}(-3x)^K + \frac{6}{5}\sum_{K=0}^{\infty}(2x)^K$,

from which we see that $R(x) = \sum_{K=0}^{\infty} r_K x^K = \sum_{K=0}^{\infty}\left(\frac{-1}{5}(-3)^K + \frac{6}{5}\cdot 2^K\right)x^K$, so

$$r_K = \frac{-1}{5}(-3)^K + \frac{6}{5}\cdot 2^K.$$

Example: Use generating functions to find a closed form for the sequence $\{s_k\}_{k=0}^{\infty}$ defined recursively by

$$\begin{cases} s_0 = 1, \\ s_k = 2s_{k-1} + 2(k+1), \quad k \geq 1. \end{cases}$$

Let $S(x) = \sum_{k=0}^{\infty} s_k x^k$, then

$$S(x) = \sum_{k=0}^{\infty} s_k x^k = s_0 + \sum_{k=1}^{\infty} s_k x^k = 1 + \sum_{k=1}^{\infty} (2s_{k-1} + 2(k+1)) x^k = 1 + 2\sum_{k=1}^{\infty} s_{k-1} x^k + 2\sum_{k=1}^{\infty} (k+1) x^k$$

$$= 1 + 2x\sum_{k=0}^{\infty} s_k x^k + 2\Big(\sum_{k=0}^{\infty} (k+1) x^k - \overset{(0+1)x^0}{1}\Big) = -1 + 2x S(x) + 2\left(\frac{1}{(1-x)^2}\right)$$

so $(1-2x)S(x) = -1 + \frac{2}{(1-x)^2} \implies S(x) = \frac{-1}{1-2x} + \frac{2}{(1-2x)(1-x)^2}$

Solving the partial fraction decomposition,

$$\frac{2}{(1-2x)(1-x)^2} = \frac{A}{1-2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2} \implies \left.\begin{aligned} A + B + C &= 2 \\ -2Ax - 3Bx - 2Cx &= 0 \\ Ax^2 + 2Bx^2 &= 0 \end{aligned}\right\} \begin{aligned} A &= 8 \\ B &= -4 \\ C &= -2, \end{aligned}$$

then $S(x) = \sum_{k=0}^{\infty} s_k x^k = \frac{-1}{1-2x} + \frac{8}{1-2x} - \frac{4}{1-x} - \frac{2}{(1-x)^2} = \frac{7}{1-2x} - \frac{4}{1-x} - \frac{2}{(1-x)^2} = 7\sum_{k=0}^{\infty}(2x)^k - 4\sum_{k=0}^{\infty} x^k - 2\sum_{k=0}^{\infty}(k+1)x^k$

So $s_k = 7 \cdot 2^k - 4 - 2(k+1) = 7 \cdot 2^k - 2k - 6$

Solving Recurrence Relations with exponential generating function (motivating the advancement operator)

Example: Solve the recurrence relation

$$\begin{cases} t_0 = 2 \\ t_k = 5t_{k-1} \end{cases}$$

With the advancement operator: Rewrite the recursion as

$0 = t_{k+1} - 5t_k = At_k - 5t_k = (A-5)t_k$, so $t_k = a \cdot 5^k$. Since

$t_0 = a \cdot 5^0 = 2$, then $a = 2$,

we get $t_k = 2 \cdot 5^k$.

With exponential generating function: Let $T(x)=\sum_{k=0}^{\infty}\frac{t_k}{k!}x^k$. Then assuming the sum converges for a positive radius,

$$T'(x)=\sum_{k=0}^{\infty}\left(\frac{t_k}{k!}x^k\right)'=\sum_{k=0}^{\infty}k\cdot\frac{t_k}{k!}x^{k-1}=\sum_{k=1}^{\infty}\frac{t_k}{(k-1)!}x^{k-1}=\sum_{k=0}^{\infty}\frac{t_{k+1}}{k!}x^k=\sum_{k=0}^{\infty}\frac{5t_k}{k!}x^k=5T(x)$$

since $0\cdot\frac{t_0}{0!}x^{0-1}=0$ $\qquad$ $\frac{k}{k!}=\frac{k}{k(k-1)!}=\frac{1}{(k-1)!}$

The solution to the differential equation $T'(x)=5T(x)$ is $T(x)=C\cdot e^{5x}$, and since $T(0)=\sum_{k=0}^{\infty}\frac{t_k}{k!}0^k=t_0+0=2=C\cdot e^0=C$, so

$$T(x)=\sum_{k=0}^{\infty}\frac{t_k}{k!}x^k=2e^{5x}=2\sum_{k=0}^{\infty}\frac{(5x)^k}{k!}=2\sum_{k=0}^{\infty}\frac{5^kx^k}{k!},\quad\text{so } t_k=2\cdot5^k.$$

The method using generating functions is simpler than the method using exponential generating functions for solving the recurrence relations seen in this course.

Newton's Binomial Theorem (The Binomial Series)

Definition: For a real number x and positive integer k, the falling factorial of x is $(x)_k := x(x-1)(x-2)\cdots(x-k+1)$.

ex. $(3)_5=3\cdot2\cdot1\cdot0\cdot(-1)=0$

$(1/3)_3=\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)$

Notice that if x is a positive integer, then $(x)_k=x(x-1)\cdots(x-k+1)=\frac{x!}{(x-k)!}=P(x,k)$

Definition: For a real number x and positive integer k, the generalized binomial coefficient is given by

$$\binom{x}{k}=\frac{(x)_k}{k!}$$

Theorem (Newton's Binomial Theorem): For any real number $p\neq0$, then

$$(1+x)^p=\sum_{k=0}^{\infty}\binom{p}{k}x^k.$$

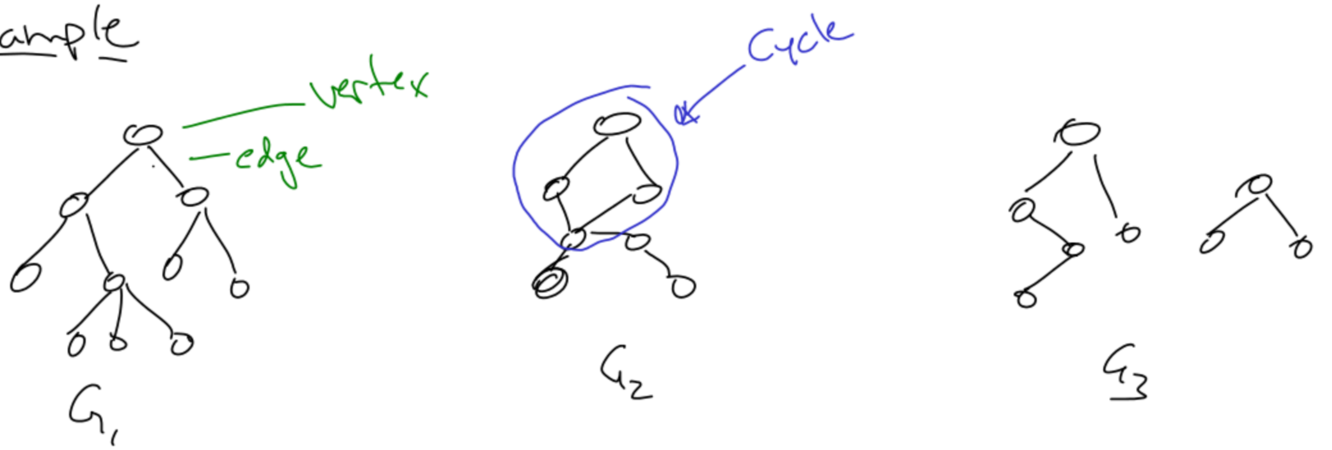
The theorem can be proved by looking at the Taylor (or McLaurin) series of $f(x)=(1+x)^p$. Since $f^{(k)}(x)=p(p-1)(p-2)\cdots(p-k+1)(1+x)^{p-k}=(p)_k(1+x)^{p-k}$, then

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=\sum_{k=0}^{\infty}\frac{(p)_k}{k!}x^k=\sum_{k=0}^{\infty}\binom{p}{k}x^k.$$

An application of recurrence relations and generating functions (leaves in rooted unlabelled binary ordered trees (RUBOTs))

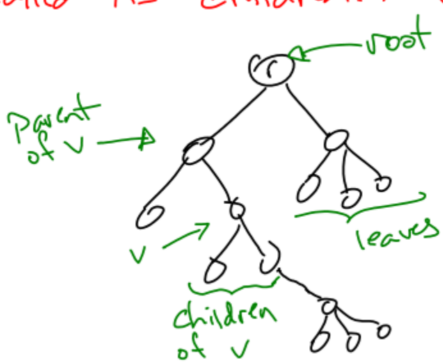
In graph theory, a <u>tree</u> is a type of graph: a collection of vertices (or nodes), with edges (or links) between them. A tree is a graph that is connected (every vertex can be reached from any other) and that has no cycles (no way to travel from a vertex, back to itself, without visiting any edge more than once); or equivalently, a connected graph with exactly one edge less than the number of vertices.

<u>Example</u>

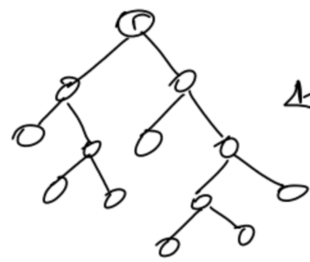


G_1, G_2, G_3 are all graphs. G_1 is the only tree. G_2 contains a cycle, and G_3 is not connected.

- A <u>rooted tree</u> is a tree with one vertex called a <u>root</u>. For every other vertex v, the vertex adjacent to v on the path from v to the root is called the <u>parent</u> of v, and all other vertices adjacent to v are called its children. A vertex with no children is a <u>leaf</u>

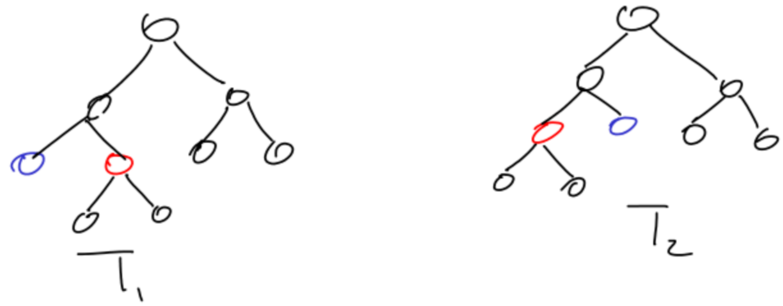


- A **binary tree** is a rooted tree such that every vertex has 0 or 2 children



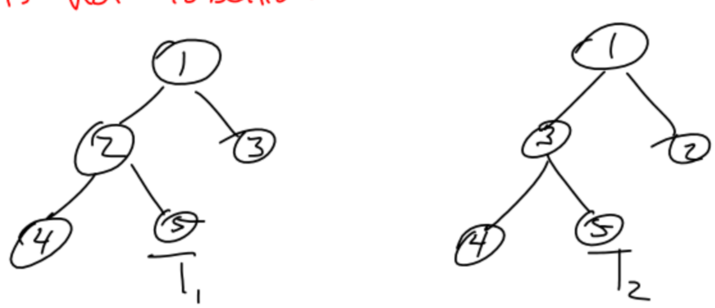
a binary tree.

- A tree is **ordered** if changing the order of the children might change the tree



If we consider T_1 and T_2 to be ordered, then they are different. If we consider them unordered, then they are the same tree.

- A tree is **labelled** if vertices are given labels, and if we consider otherwise identical trees to be different if the labels differ. A tree is **unlabelled** if it is not labelled.

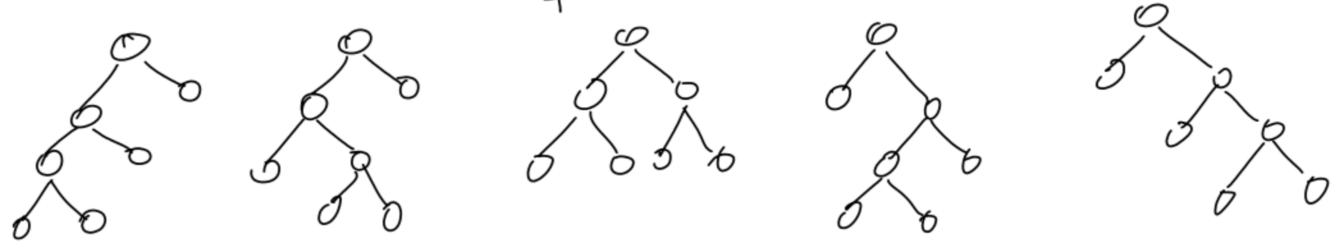


If T_1 and T_2 are labelled, then they are different. If they are unlabelled, then they are the same tree.

Let c_n be the number of rooted unlabelled binary ordered trees (RUBOTs) with exactly n leaves.

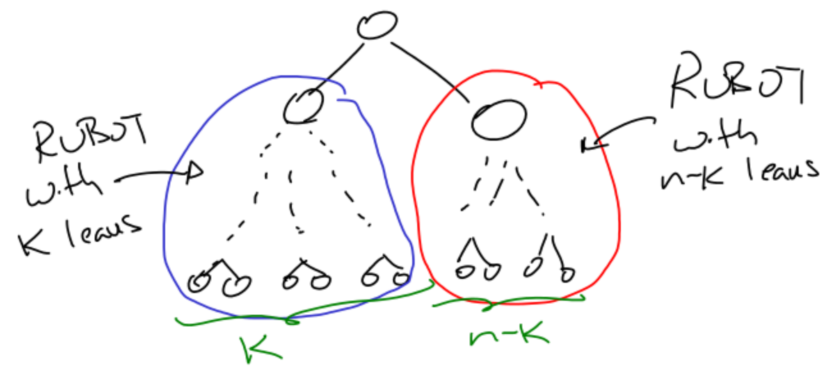
$C_0=0$. $C_1=1$ $C_2=1$ $C_3=2$

$C_4=5$



What is C_n? If we remove the root, we have 2 new RUBOTs, the left tree with k leaves, and the right tree with $n-k$ leaves.

So to construct a RUBOT with n leaves, take 2 RUBOTs; one with k leaves, the other with $n-k$ leaves, and attach the roots of both to a new root vertex.



So for $n \geq 2$,

$$C_n = \sum_{k=1}^{n-1} C_k C_{n-k} = \sum_{k=0}^{n} C_k C_{n-k}$$

this is fine since $C_0 = 0$

Recall that for $F(x) = \sum_{n=0}^{\infty} f_n x^n$ $G(x) = \sum_{n=0}^{\infty} g_n x^n$ then

$H(x) = F(x)G(x) = \sum_{n=0}^{\infty} h_n x^n$, where $h_n = \sum_{k=0}^{n} f_k g_{n-k}$.

So if $C(x) = \sum_{n=0}^{\infty} C_n x^n$, then

since $\sum_{k=0}^{0} C_0 C_{0-0} x^0 = 0$

$\sum_{k=0}^{1} C_k C_{1-k} x = C_0 C_1 x + C_1 C_0 x = 0$

$$C(x) = C_0 + C_1 x + \sum_{n=2}^{\infty} C_n x^n = 0 + x + \sum_{n=2}^{\infty} \sum_{k=0}^{n} C_k C_{n-k} x^k = 0 + x + \sum_{n=0}^{\infty} \sum_{k=0}^{n} C_k C_{n-k} x^k$$

$$= x + (C(x))^2.$$

Therefore $C(x) = x + C^2(x) \Rightarrow C^2(x) - C(x) + x = 0 \Leftrightarrow C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2}$

$$= \frac{1}{2} \pm \frac{1}{2}(1-4x)^{1/2}$$

Using Newton's Binomial Theorem, $(1-4x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n$.

Let's look at $\binom{1/2}{n}$ more closely; for $n \geq 1$

$$\binom{1/2}{n} = \frac{(1/2)_n}{n!} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\cdots\left(\frac{1}{2}-n+1\right)}{n!} = \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\cdots\cdots\left(-\frac{(2n-3)}{2}\right)$$

$$= \frac{(-1)^{n-1} 1\cdot 3\cdot 5\cdot \cdots\cdot (2n-3)}{n!\,2^n} = \frac{(-1)^{n-1} 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 \cdots \cdot (2n-3)(2n-2)}{n!\,2^n\cdot 2\cdot 4\cdot 6\cdots\cdots (2n-2)}$$

$$= \frac{(-1)^{n-1}(2n-2)!}{n!\,2^n\cdot 2^{n-1}(n-1)!} = \frac{(-1)^{n-1}}{n2^{2n-1}} \cdot \frac{(2n-2)!}{(n-1)!(n-1)!} = \frac{(-1)^{n-1}}{n2^{2n-1}} \cdot \binom{2n-2}{n-1}$$

So $C(x) = \frac{1}{2} \pm \frac{1}{2} \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n = \frac{1}{2} \pm \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^{2n-1}} \binom{2n-2}{n-1} (-4x)^n\right)$

$$= \frac{1}{2} \pm \frac{1}{2} \pm \sum_{n=1}^{\infty} \frac{\overbrace{(-1)^{n-1}(-1)^n}^{=(-1)^{2n-1}=-1}}{n\,\cancel{2^{2n}}} \cancel{(2^2)^n} \binom{2n-2}{n-1} = \frac{1}{2} \pm \frac{1}{2} \mp \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n$$

Since $c_n \geq 0$ for all n, we take the "minus option" for $\frac{1 \pm \sqrt{1-4x}}{2}$, so

$$C(x) = \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n = \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n = \sum_{n=0}^{\infty} c_n x^n, \text{ so}$$

$$c_0 = 0,\quad c_n = \frac{1}{n}\binom{2n-2}{n-1}$$

C_{n-1}

Catalan numbers.

Generating Functions and Recurrence Relations

1.1 Generating Functions

Exercise 1.1.1. Find a closed form of the generating function for the sequence $\{a_k | k \geq 0\}$ given by

(a) $a_k = 3^k$

(b) $a_k = \frac{1}{3^k}$

(c) $a_k = \begin{cases} 0 & k = 0, 1, 2, 3, 4 \\ k - 4 & k \geq 5 \end{cases}$

(d) $a_k = \begin{cases} 2^k & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

Solution. (a)

$$\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} (3x)^k = \frac{1}{1-3x}$$

(b)

$$\sum_{k=0}^{\infty} \frac{1}{3^k} x^k = \sum_{k=0}^{\infty} \left(\frac{x}{3}\right)^k = \frac{1}{1-x/3} = \frac{3}{3-x}$$

(c)

$$\sum_{k=5}^{\infty} (k-4)x^k = \sum_{k=0}^{\infty} (k+1)x^{k+5} = x^5 \sum_{k=0}^{\infty} (k+1)x^k = x^5 \sum_{k=0}^{\infty} \binom{k+1}{1} x^k = \frac{x^5}{(1-x)^2}$$

(d)

$$\sum_{m \text{even}} 2^m x^m = \sum_{k=0}^{\infty} 2^{2k} x^{2k} = \sum_{k=0}^{\infty} ((2x)^2)^k = \frac{1}{1-(2x)^2} = \frac{1}{1-4x^2}$$

□

Exercise 1.1.2. what is the coefficient of x^{12} in

(a) $(x^2 + x^4 + x^6)(x^6 + x^8 + x^{10})$

(b) $\frac{x^7}{1-(x/2)^2}$

(c) $\frac{1-x^{22}}{1-x}$

(d) $\frac{x^2}{(1-2x^2)^4}$

Solution. (a)

$$(x^2+x^4+x^6)(x^6+x^8+x^{10}) = x^8+x^{10}+x^{12}+x^{10}+x^{12}+x^{14}+x^{12}+x^{14}+x^{16} = x^8+2x^{10}+3x^{12}+2x^{14}+x^{16},$$

so the coefficient of x^{12} is 3.

(b)

$$\frac{x^7}{1-(x/2)^2} = x^7 \sum_{k=0}^{\infty} \left(\left(\frac{x}{2} \right)^2 \right)^k = x^7 \sum_{k=0}^{\infty} \frac{1}{2^{2k}} x^{2k} = \sum_{k=0}^{\infty} \frac{1}{2^{2k}} x^{2k+7}.$$

Since $12 = 2k+7$ never occurs as an exponent for 7 for any integer k, the coefficient of x^{12} is 0.

(c)

$$\frac{1-x^{22}}{1-x} = 1 + x + x^2 + \cdots + x^21,$$

so the coefficient of x^{12} is 1.

(d)

$$\frac{x^2}{(1-2x^2)^4} = x^2 \sum_{k=0}^{\infty} \binom{4+k-1}{4-1} (2x^2)^k = x^2 \sum_{k=0}^{\infty} \binom{k+3}{3} 2^k x^2 k = \sum_{k=0}^{\infty} \binom{k+3}{3} 2^k x^{2k+2},$$

and $2k+2 = 12$ for $k = 5$, so the coefficient of x^{12} is $\binom{8}{3}2^5$.

$\square$

Exercise 1.1.3. What is the number of integer solutions to $x_1+x_2+x_3 = k$ with the restrictions $0 \le x_1 \le 3$, x_2 must be a multiple of 4, and $x_3 \ge 1$?

Solution. The generating function for the number of solutions to $x_1 = k$ with $0 \le x_1 \le 3$ is

$$1 + x + x^2 + x^3 = \frac{1-x^4}{1-x}.$$

The generating function for the number of solutions to $x_2 = k$ with x_2 being a multiple of 4 is

$$1 + x^4 + x^8 + x^{12} + \cdots = \sum_{k=0}^{\infty} (x^4)^k = \frac{1}{1-x^4}.$$

The generating function for the number of solutions to $x_3 = k$ with $x_3 \ge 1$ is

$$x + x^2 + x^3 + x^4 + \cdots = x \sum_{k=0}^{\infty} x^k = \frac{x}{1-x}.$$

Therefore, the generating functions for the number of solutions to $x_1 + x_2 + x_3 = k$ with the conditions set above is

$$\left(\frac{1-x^4}{1-x} \right) \left(\frac{1}{1-x^4} \right) \left(\frac{x}{1-x} \right) = \frac{x}{(1-x)^2} = x \sum_{k=0}^{\infty} \binom{2+k-1}{2-1} x^k = \sum_{k=0}^{\infty} \binom{k+1}{1} x^{k+1} = \sum_{k=1}^{\infty} \binom{k}{1} x^k,$$

whose k'th coefficient is $\binom{k}{1} = k$. So the number of solutions to $x_1 + x_2 + x_3 = k$ with the conditions set above is k. $\square$

Exercise 1.1.4. Use generating functions to find the number of ways can 24 apples be distributed amongst 4 students so that every student gets at least 3 apples, but no more than 8?

Solution. For each student, their generating function for the number of ways of distributing k apples to that student such that the number of apples received is between 3 and 8 is given by

$$x^3 + x^4 + x^5 + x^6 + x^7 + x^8 = x^3\left(1 + x + x^2 + x^3 + x^4 + x^5\right),$$

so the generating function for the number of distributions amongst 4 students is

$$\left(x^3\left(1 + x + x^2 + x^3 + x^4 + x^5\right)\right)^4 = x^{12}\left(1 + x + x^2 + x^3 + x^4 + x^5\right)^4 = x^{12}\left(\frac{1 - x^6}{1 - x}\right)^4 = x^{12}(1-x^6)^4\frac{1}{(1 - x)^4}$$

so we need to find the coefficient of x^12 in $(1 - x^6)^4\frac{1}{(1-x)^4}$. Let

$$F(x) = (1 - x^6)^4 = \sum_{k=0}^{\infty} a_k x^k$$

and

$$G(x) = \frac{1}{(1 - x)^4} = \sum_{k=0}^{\infty} b_k x^k.$$

From the Binomial Theorem,

$$F(x) = \sum_{k=0}^{4} \binom{4}{k}(-x^6)^k = 1 - \binom{4}{1}x^6 + \binom{4}{2}x^{12} - \binom{4}{3}x^{18} + \binom{4}{4}x^{24},$$

and we know that $b_k = \binom{4+k-1}{4-1} = \binom{3+k}{3}$. Therefore, the coefficient of x^{12} in $F(x)G(x)$ is given by

$$\sum_{j=0}^{12} a_j b_{12-j} = a_1 b_{12} + a_6 b_6 + a_{12} b_0 = 1 \cdot \binom{15}{3} - \binom{4}{1}\binom{9}{3} + \binom{4}{2} \cdot 1,$$

which is also the number of ways of distributing the 24 apples such that every students receives between 3 and 8 apples. □

Exercise 1.1.5. How many strings of length n are there consisting of $\{a, b, c, d\}$ so that the number of b's is even, the number of c's is odd, and d appears at least once.

Solution. The exponential generating function for the number of strings of length k consisting of a's is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = e^x.$$

The exponential generating fucntion for the number of strings of length k consisting of an even number of b's is

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{e^x + e^{-x}}{2}.$$

The exponential generating function for the number of strings of length k consisting of an odd number of c's is

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{e^x - e^{-x}}{2}.$$

The exponential generating function for the number of strings of length $k \geq 1$ consisting of d's is

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = e^x - 1.$$

So the exponential generating function for the number of strings satisfying the conditions above is given by

$$e^x\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)(e^x-1)=e^x\left(\frac{e^{2x}-e^{-2x}}{4}\right)(e^x-1)=\frac{e^{4x}-1}{4}-\frac{e^{3x}-e^{-x}}{4}.$$

The required exponential generating functions are given by

$$e^{4x}-1=\sum_{k=0}^{\infty}\frac{(4x)^k}{k!}-1=\sum_{k=1}^{\infty}4^k\frac{x^k}{k!}$$

and

$$e^{3x}-e^{-x}=\sum_{k=0}^{\infty}\frac{(3x)^k}{k!}-\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}=\sum_{k=0}^{\infty}(3^k-(-1)^k)\frac{x^k}{k!},.$$

So the number of strings we are looking for is given by

$$\frac{1}{4}\left(4^k-3^k+(-1)^k\right)$$

when $k\geq 1$, and

$$\frac{1}{4}\left(-3^0+(-1)^0\right)=0$$

for $k=0$. $\square$

1.2 Recurrence Relations

Exercise 1.2.1. Solve the recurrence equation $f_0 = 2$, $f_1 = 5$, $f_k = 6f_{k-1} - 8f_{k-2}$, $k \geq 2$.

Solution. We rewrite the recursion as

$$0 = f_{k+2} - 6f_{k+1} + 8f_k = A^2 f_k - 6Af_k + 8f_k = (A^2 - 6A + 8)f_k = (A-2)(A-4)f_k,$$

and so we conclude $f_k = a_1 2^k + a_2 4^k$. Since

$$f_0 = a_1 + a_2 = 2$$

and

$$f_1 = 2a_1 + 4a_2 = 5,$$

we solve the above system to get $a_1 = 3/2$ and $a_2 = 1/2$. Therefore,

$$f_k = \frac{3}{2}2^k + \frac{1}{2}4^k.$$

□

Exercise 1.2.2. Solve the recurrence equation $g_0 = -1$, $g_k = 3g_{k-1} + (-1)^k + 1$, $k \geq 2$.

Solution. We rewrite the recursion as

$$(-1)^{k+1} + 1 = g_{k+1} - 3g_k = Ag_k - 3g_k = (A-3)g_k.$$

Solving the homogenous equation $(A-3)g_k' = 0$ yields $g_k' = a3^k$ for some constant a. Now to find a particular solution to $(-1)^{k+1} + 1 = (A-3)g_k''$. A good guess is $g_k'' = b(-1)^{k+1} + c$ for some b and c. Then

$$\begin{aligned}
(-1)^{k+1} + 1 &= (A-3)g_k'' \\
&= (A-3)(b(-1)^{k+1} + c) \\
&= A(b(-1)^{k+1} + c) - 3(b(-1)^{k+1} + c) \\
&= b(-1)^{k+2} + c - 3b(-1)^{k+1} - 3c \\
&= b(-1)(-1)^{k+1} - 3b(-1)^{k+1} - 2c \\
&= -4b(-1)^{k+1} - 2c,
\end{aligned}$$

so $b = -1/4$ and $c = -1/2$. Therefore

$$g_k'' = \frac{-1}{4}(-1)^{k+1} - \frac{1}{2} = \frac{1}{4}(-1)^k - \frac{1}{2}$$

is a particular solution, and so

$$g_k = g_k' + g_k'' = a3^k + \frac{1}{4}(-1)^k - \frac{1}{2}.$$

Since

$$g_0 = a + \frac{1}{4} - \frac{1}{2} = -1,$$

then $a = -3/4$, and so we get that

$$g_k = \frac{-3}{4} \cdot 3^k - \frac{1}{4}(-1)^k - \frac{1}{2}.$$

□

Exercise 1.2.3. Solve the recurrence equation $h_0 = 5$, $h_1 = 6$, $h_k = 4h_{k-1} - 4h_{k-2} + k - 1$, $k \geq 2$.

Solution. We rewrite the recursion as

$$k+1=(k+2)-1=h_{k+2}-4h_{k+1}+4h_k=A^2h_k-4Ah_k+4h_k=(A^2-4A+4)h_k=(A-2)^2h_k.$$

Solving the homogoenous equation $(A-2)^2h_k=0$ yields $h_k'=a_12^k+ka_22^k$. Now to find a particular solution $k+1=(A-2)^2h_k''$. A good guess is $h_k''=bk+c$ for some b and c. Then

$$\begin{aligned}k+1&=(A-2)^2h_k''\\&=(A^2-4A+4)(bk+c)\\&=A^2(bk+c)-4A(bk+c)+4(bk+c)\\&=b(k+2)+c-4b(k+1)-4c+4bk+4c\\&=bk-2b+c,\end{aligned}$$

so $b=1$ and $-2b+c=1\Rightarrow c=3$. So $h_k''=k+3$ is a particular solution. So

$$h_k=h_k'+h_k''=a_12^k+ka_22^k+k+3.$$

Then since $h_0=a_1+3=5$ and

$$h_1=2a_1+2a_2+1+3=6,$$

we get $a_1=2$ and $a_1=-1$. So

$$h_k=2\cdot 2^k=k\cdot 2^k+k+3=(2-k)2^k+k+3.$$

□

Exercise 1.2.4. How many binary strings of length n have no occurrences of 110?

Solution. Let s_k be the number of binary strings of length k without occurences of 110. There is one string of lenght 0 (the empty string), there are 2 strings of length 1, and $2^2=4$ strings of length 2. There are a total of 2^8 strings of length 3, but only 7 of them that are not 110. Therefore, $s_0=1, s_1=2, s_2=4$ and $s_3=7$.

To create a longer string of length $k\geq 4$ without occurences of 110, we can take a 110-avoiding string of length $k-1$ and add either a 1 or a 0, there are 2 ways of doing this. However, we may have added strings of length $k-1$ that start with 10 and added a 1, so these we need to remove. There are then s_{k-3} such strings to remove: take a string of length $k-3$ that avoids 110 and add 10 at the beginning.

Therefore, we get that $s_k=2s_{k-1}-s_{k-3}$ for $k\geq 4$. Now we solve the recursion. Using advancement operator, the recursion can be rewritten as

$$0=s_{k-3}-2s_{k+2}+s_k=A^3s_k-2A^2s_k+s_k=(A^3-2A^2+1)s_k=(A-1)(A^2-A-1)s_k.$$

The roots of A^2-A-1 are $\frac{1\pm\sqrt{5}}{2}$. Then

$$s_k=a_1\cdot 1^k+a_2\left(\frac{1+\sqrt{5}}{2}\right)^k+a_3\left(\frac{1-\sqrt{5}}{2}\right)^k.$$

Since

$$\begin{aligned}&s_0=a_1+a_2+a_3=1\\&s_1=a_1+a_2\left(\frac{1+\sqrt{5}}{2}\right)+a_3\left(\frac{1-\sqrt{5}}{2}\right)=2\\&s_2=a_1+a_2\left(\frac{1+\sqrt{5}}{2}\right)^2+a_3\left(\frac{1-\sqrt{5}}{2}\right)^2=a_1+a_2\left(\frac{3+\sqrt{5}}{2}\right)+a_3\left(\frac{3-\sqrt{5}}{2}\right)=4.\end{aligned}$$

To solve this system of equations, notice that $s_2 - s_1$ yields $a_2 + a_3 = 2$. So $s_2 - s_1 - s_0$ gives $a_1 = -1$. Next, replacing a_1 with 1 and $a_3 = 2 - a_2$ in s_1 gives

$$1 + a_2\left(\frac{1+\sqrt{5}}{2}\right) + (2 - a_2)\left(\frac{1-\sqrt{5}}{2}\right) = 2$$

which simplifies to

$$\sqrt{5}a_2 = 2 + \sqrt{5} \Rightarrow a_2 = \frac{2+\sqrt{5}}{\sqrt{5}} = 1 + \frac{2\sqrt{5}}{5}.$$

Then since $a_3 = 2 - a_2$,

$$a_3 = 1 - \frac{2\sqrt{5}}{5}.$$

Therefore, we get that the number of binary strings of length k with no occurences of 110 is given by

$$s_k = -1 + \left(1 + \frac{2\sqrt{5}}{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(1 - \frac{2\sqrt{5}}{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^k.$$

□

Exercise 1.2.5. Recall the k'th Fibonacci number f_k defined recursively by $f_0 = 0, f_1 = 1$, and $f_k = f_{k-1} + f_{k-2}$ for $k \geq 2$. Find a closed form for f_k.

Solution. We rewrite the recursion as

$$0 = f_{k+2} - f_{k+1} - f_k = A^2 f_k - Af_k - f_k = (A^2 - A - 1)f_k = \left(A - \frac{1+\sqrt{5}}{2}\right)\left(A - \frac{1-\sqrt{5}}{2}\right) f_k,$$

so

$$f_k = a_1\left(\frac{1+\sqrt{5}}{2}\right)^k + a_2\left(\frac{1-\sqrt{5}}{2}\right)^k.$$

Since

$$\begin{aligned} f_0 &= a_1 + a_2 = 0 \\ f_1 &= a_1\left(\frac{1+\sqrt{5}}{2}\right) + a_2\left(\frac{1-\sqrt{5}}{2}\right) = 1, \end{aligned}$$

so we get that $a_1 = 1/\sqrt{5}$ and $a_2 = -1/\sqrt{5}$, so

$$f_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k.$$

□

1.3 Further Examples

Exercise 1.3.1. Solve exercise 1.2.1 using generating functions.

Solution. Let $F(x) = \sum_{k=0}^{\infty} f_k x^k$, which we rewrite as

$$\begin{aligned} F(x) &= f_0 + f_1 x + \sum_{k=2}^{\infty} f_k x^k \\ &= 2 + fx + \sum_{k=2}^{\infty} (6f_{k-1} - 8f_{k-2}) x^k \\ &= 2 + 5x + 6\sum_{k=2}^{\infty} f_{k-1}x^k - 8\sum_{k=2}^{\infty} f_{k-2}x^k \\ &= 2 + 5x + 6\left(\sum_{k=0}^{\infty} f_k x^{k+1} - f_0 x\right) - 8\sum_{k=0}^{\infty} f_k x^{k+2} \\ &= 2 + 5x + 6(xF(x) - 2x) - 8x^2F(x) \\ &= 2 - 7x + 6xF(x) - 8x^2F(x). \end{aligned}$$

So $F(x) - 6xF(x) + 8x^2F(x) = (1 - 6x + 8x^2)F(x) = 2 - 7x$, which we rearrange as

$$F(x) = \frac{2-7x}{1-6x+8x^2} = \frac{2-7x}{(1-4x)(1-2x)} = \frac{A}{1-4x} + \frac{B}{1-2x}.$$

Solving the partial fraction decomposition

$$A - 2Ax + b - 4Bx = 2 - 7x \Rightarrow \begin{cases} A + B = 2, \\ -2A - 4B = -7, \end{cases}$$

gives $A = 1/2$ and $B = 3/2$, so

$$F(x) = \frac{1}{2}\left(\frac{1}{1-4x}\right) + \frac{3}{2}\left(\frac{1}{1-2x}\right) = \frac{1}{2}\sum_{k=0}^{\infty}(4x)^k + \frac{3}{2}\sum_{k=0}^{\infty}(2x)^k,$$

so

$$f_k = \frac{1}{2}4^k + \frac{3}{2}2^k.$$

□

Exercise 1.3.2. Solve exercise 1.2.2 using generating functions.

Solution. Let $G(x) = \sum_{k=0}^{\infty} g_k x^k$, which we can rewrite as

$$\begin{aligned} G(x) &= g_0 + \sum_{k=1}^{\infty} g_k x^k \\ &= -1 + \sum_{k=1}^{\infty} (3g_{k-1} + (-1)^k + 1) x^k \\ &= -1 + 3\sum_{k=1}^{\infty} g_{k-1}x^k + \sum_{k=1}^{\infty} ((-1)^k + 1)x^k \\ &= -1 + 3\sum_{k=0}^{\infty} g_k x^{k+1} + \sum_{k=0}^{\infty} ((-1)^k + 1)x^k - ((-1)^0 + 1)x^0. \end{aligned}$$

From here, we can split

$$\sum_{k=0}^{\infty}((-1)^k+1)x^k = \sum_{k=0}^{\infty}(-1)^k x^k + \sum_{k=0}^{\infty} x^k = \frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2},$$

or notice that

$$(-1)^k+1) = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

so

$$\sum_{k=0}^{\infty}((-1)^k+1)x^k = \sum_{k=0}^{\infty} 2x^{2k} = 2\sum_{k=0}^{\infty} x^{2k} = 2\left(\frac{1}{1-x^2}\right).$$

Either way, we continue our derivation (with replacing $((-1)^0+1)x^0 = 2$)

$$\begin{aligned} G(x) &= -1 + 3\sum_{k=0}^{\infty} g_k x^{k+1} + \sum_{k=0}^{\infty}((-1)^k+1)x^k - 2 \\ &= -3 + 3xG(x) + \frac{2}{1-x^2}. \end{aligned}$$

Therefore,

$$G(x) - 3xG(x) = (1-3x)G(x) = -3 + \frac{2}{1-x^2},$$

so

$$G(x) = \frac{-3}{1-3x} + \frac{2}{(1-3x)(1-x^2)} = \frac{-3}{1-3x} + \frac{2}{(1-3x)(1-x)(1+x)}.$$

Solving the partial fraction decomposition

$$\frac{2}{(1-3x)(1-x)(1+x)} = \frac{A}{1-3x} + \frac{B}{1-x} + \frac{C}{1+x} \Rightarrow \begin{cases} A+B+C & = 2 \\ -2Bx - 4Cx & = 0 \\ -Ax^2 - 3Bx^2 + 3Cx^2 & = 0 \end{cases}$$

Solving the above system of equations gives $A = 9/4, B = -1/2, C = 1/4$, so

$$\begin{aligned} G(x) &= \frac{-3}{1-3x} + \frac{9}{4}\left(\frac{1}{1-3x}\right) - \frac{1}{2}\left(\frac{1}{1-x}\right) + \frac{1}{4}\left(\frac{1}{1+x}\right) \\ &= \frac{-3}{4}\left(\frac{1}{1-3x}\right) - \frac{1}{2}\left(\frac{1}{1-x}\right) + \frac{1}{4}\left(\frac{1}{1+x}\right) \\ &= \frac{-3}{4}\sum_{k=0}^{\infty}(3x)^k - \frac{1}{2}\sum_{k=0}^{\infty} x^k + \frac{1}{4}\sum_{k=0}^{\infty}(-x)^k, \end{aligned}$$

so

$$g_k = \frac{-3}{4} \cdot 3^k - \frac{1}{2} + \frac{1}{4} \cdot (-1)^k.$$

□

Exercise 1.3.3. Solve exercise 1.2.3 using generating functions.

Solution. Let $H(x) = \sum_{k=0}^{\infty} h_k x^k$, which we rewrite as

$$\begin{aligned}
H(x) &= h_0 + h_1 x + \sum_{k=2}^{\infty} h_k x^k \\
&= 5 + 6x + \sum_{k=2}^{\infty} (4h_{k-1} - 4h_{k-2} + k - 1)x^k \\
&= 5 + 6x + 4\sum_{k=2}^{\infty} h_{k-1}x^k - 4\sum_{k=2}^{\infty} h_{k-2}x^k + \sum_{k=2}^{\infty}(k-1)x^k \\
&= 5 + 6x + 4\left(\sum_{h=0}^{\infty} h_k x^{k+1} - h_0 x\right) - 4\sum_{k=0}^{\infty} h_k x^{k+2} + \sum_{k=0}^{\infty} x^{k+2} \\
&= 5 + 6x + 4xH(x) - 20x - 4x^2 H(x) + x^2 \sum_{k=0}^{\infty}(k+1)x^k \\
&= 5 - 14x + 4xH(x) - 4x^2 H(x) + \frac{x^2}{(1-x)^2}.
\end{aligned}$$

Therefore,

$$H(x) - 4xH(x) + 4x^2 H(x) = (1 - 4x + 4x^2)H(x) = 5 - 14x + \frac{x^2}{(1-x)^2},$$

so

$$H(x) = \frac{5 - 14x}{1 - 4x + 4x^2} + \frac{x^2}{(1-x)^2(1 - 4x + 4x^2)} = \frac{5 - 14x}{(1-2x)^2} + \frac{x^2}{(1-x)^2(1-2x)^2}.$$

We know that

$$\frac{1}{(1-2x)^2} = \sum_{k=0}^{\infty}(k+1)(2x)^k = \sum_{k=0}^{\infty}(k+1)2^k x^k,$$

so

$$\begin{aligned}
\frac{5 - 14x}{(1-2x)^2} &= 5\sum_{k=0}^{\infty}(k+1)2^k x^k - 14x\sum_{k=0}^{\infty}(k+1)2^k x^k \\
&= 5\sum_{k=0}^{\infty}(k+1)2^k x^k - 7\sum_{k=0}^{\infty}(k+1)2^{k+1} x^{k+1} \\
&= 5\sum_{k=0}^{\infty}(k+1)2^k x^k - 7\sum_{k=1}^{\infty} k2^k x^k \\
&= 5\sum_{k=0}^{\infty}(k+1)2^k x^k - 7\sum_{k=0}^{\infty} k2^k x^k,
\end{aligned}$$

the last line follows since $k2^k x^k = 0$ when $k = 0$. Next, we use partial fraction decomposition on the remaining fraction,

$$\frac{x^2}{(1-x)^2(1-2x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-2x)} + \frac{D}{(1-2x)^2}.$$

Multiplying everything out gives

$$\begin{aligned}
x^2 &= A(1-x)(1-2x)^2 + B(1-2x)^2 + C(1-x)^2(1-2x) + D(1-x)^2 \\
&= A - 5Ax + 8Ax^2 - 4Ax^3 + B - 4Bx + 4Bx^2 + C - 4Cx + 5Cx^2 - 2Cx^3 + D - 2Dx + Dx^2,
\end{aligned}$$

which gives the system of equations

$$\begin{aligned} A + B + C + D &= 0 \\ -5Ax - 4Bx - 4Cx - 2Dx &= 0 \\ 8Ax^2 + 4Bx^2 + 5Cx^2 + Dx^2 &= x^2 \\ -4Ax^3 - 2Cx^3 &= 0, \end{aligned}$$

which has the solution $A = 2, B = 1, C = -4, D = 1$. Therefore,

$$\begin{aligned} H(x) &= \frac{5 - 14x}{(1-2x)^2} + \frac{x^2}{(1-x)^2(1-2x)^2} \\ &= \frac{5}{(1-2x)^2} - \frac{14x}{(1-2x)^2} + \frac{2}{1-x} + \frac{1}{(1-x)^2} - \frac{4}{1-2x} + \frac{1}{(1-2x)^2} \\ &= 5\sum_{k=0}^{\infty}(k+1)2^k x^k - 7\sum_{k=0}^{\infty} k2^k x^k + 2\sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty}(k+1)x^k - 4\sum_{k=0}^{\infty} 2^k x^k + \sum_{k=0}^{\infty}(k+1)2^k x^k, \end{aligned}$$

so

$$\begin{aligned} h_k &= 5(k+1)2^k - 7k2^k + 2 + (k+1) - 4 \cdot 2^k + (k+1)2^k \\ &= -k2^k + 2 \cdot 2^k + 3 \\ &= (2-k)2^k + 3. \end{aligned}$$

$\square$